AN ANALYTICAL TECHNIQUE FOR THE DESIGN OF MULTI-LOOP SERVOMECHANISM COMPENSATION

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ABSTRACT

This thesis presents an analytical technique for the design of compensation for multi-loop servomechanisms. A completely general multi-loop system is represented in a standard block diagram. For this system a standard determinant is formed which uniquely represents each block, loop and path in the block diagram. Based on this determinant, the effect of all possible compensators on the coefficients of the system characteristic equation is found. This information is presented in a Location Matrix. A model characteristic equation is synthesized from the static and dynamic performance specifications. Comparison of the uncompensated characteristic equation to the proper model equation shows which coefficients must be altered. The Location Matrix will show all the compensator-path combinations which will

affect the desired change. The method requires no intuition, and so can be applied by persons without extensive training in the servomechanism field. A step-by-step procedure for application of this technique is included as a summary.

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TABLE OF CONTENTS

OBJECT

The object of this thesis is to present an analytical technique for the design of multi-loop servomechanism compensation. This technique is set forth under the following topics:

(a) Formation of the Characteristic Equation.

(b) The Effect of Performance Specifications on the Coefficients of the Characteristic Equation.

(c) The Effect of Compensation on the Coefficients of the Characteristic Equation.

(d) The Choice of Compensation.

Chapter 1

INTRODUCTION and METHOD OF APPROACH

1.1 Introduction. The problem of compensation design for a multi-loop servomechanism is not a new one. Indeed, most standard servomechanisms or control systems texts give it some mention. Most such works treat the problem as an extension of the single-loop problem, and attempt to apply single-loop concepts to its solution.

The penalties of such an approach are numerous and serious. The first such penalty arises from the technique of getting the system into a single loop configuration. This involves a block diagram reduction or some similar procedure. As a result of this reduction, the identities of the original blocks, and the physical components they represent, are lost. Far more serious than this, in a compensation design problem, is the fact that the information about the arrangement of the components is lost. The analysis of this reduced system would show algebraically what kind of compensation was needed. It would be impossible to transform this information into a given compensator fed around certain main path elements. The whole sense of element and path configuration no longer has any correlation with the algebra.

There are two ways out of this dilemma. The designer can insert all possible compensators, in all usable paths,

and re-do the analysis until he finds one combination that gives the desired result. Obviously this is a monumental task for any system of reasonable complexity. There are graphical techniques which eliminate some of this work, but they still involve trial and error. The alternative is the intuitive approach. Here the designer must have such vast experience that he generally knows the effect of all compensators, and so can eliminate from consideration many which could not work.

In this thesis an analytical technique is presented which requires no intuition. It examines the effect of all possible compensators without the accompanying drudgery.

1.2 Method of Approach

A general multi-loop control system, consisting of all possible direct, feedback and feed forward paths, is formulated. Its components are presumed to be describable by ordinary, linear integro-differential equations, with constant coefficients. When inputs to this system are mentioned, the terms "step", "ramp", and "parabola" of input will be used. A step of the input quantity is that input which yields zero error when there is unity feedback from output to input. For a Type I positional servo, a step would be a step of position. For a Type II velocity servo, the step would be a step of velocity, etc. By treating such a general system, the ensuing results can be applied to nearly all practical systems.

This analytical compensation design technique uses as its cornerstone the characteristic equation of the system. After a brief discussion of the methods for choosing direct path

components, a method for forming this characteristic equation is described. The system is represented in a standard block diagram. Using circuit analysis techniques, the interaction of the blocks, path and loops in this diagram is shown in a characteristic determinant of the system. It is from this determinant that the characteristic equation is written.

The next step in the procedure relates performance specifications to the coefficients of the characteristic equation. Static specifications such as "following error" and output in response to a load torque, place limits on coefficients or on ratios of coefficients in the characteristic equation. Because of the vast amount of work which has been done on simple second order systems, most dynamic specifications are expressible in terms of the parameters of such systems. The compensated system is made to have a response not significantly different from the specified second order response. This is done by making the quadratic characteristic equation of the second order system a factor of a model characteristic equation for the compensated system. All other roots of this model equation are forced to be so large that the second order roots dominate the response.

The effect of all possible compensators on the coefficients of the characteristic equation is found through the use of the characteristic determinant. All possible paths in the system are uniquely represented in this determinant. Closing any of these paths with a compensator adds a cofactor to the existing determinant. A rapid method for determining

the effect of this path closing on the coefficients of the characteristic equation is presented. This information is displayed in a Location Matrix.

At this stage of the design process, the designer has a model characteristic equation representing the specifications, and an actual characteristic equation representing the uncompensated system. A comparison of these shows which coefficients must be altered by compensation. The Location Matrix indicates what compensator, in what location, will effect the required change.

This thesis confines itself to the problem of assuring dominance of a desired second order response. The technique itself is not so limited, and could be used to force a system to perform like any desired third or higher order system.

CHAPTER 2

BASIC DESIGN CONSIDERATIONS

1. Introduction

Some servomechanisms perform a computing function. Examples of this type of servo are found in fire control systems. In these servos, the output or outputs may be complicated mathematical functions of several inputs. The mathematical relationships to be instrumented will dictate the choice of equipment to be used. The type of servo of primary interest in this thesis, however, is one which drives a load so that its action duplicates the action represented by the input signal. An example of this type of servo is the drive system for a gun. Its input is a voltage or shaft rotation which represents the desired orientation of the gun barrel. It is the function of the servo to control the power that drives the gun mount so that the desired gun barrel orientation is achieved. The preliminary choice of equipment to perform this type of function is considered in this chapter. These considerations result in a basic system with sufficient power and accuracy to meet the fundamental requirements set by the specifications. Subsequent chapters deal with modifying the basic design to meet specifications concerning the more detailed nature of the servo's response.

2. Components of a Typical Servomechanism

Figure II-1 is a diagram of a typical servomechanism. Its purpose might be to control the angular position and/or velocity of a shaft. The shaft is represented by the block labeled "Load". The input or command might be the angular position of a shaft from a computer. The command receiver and the output measuring device convert the input and output into quantities which can be easily compared. The comparator subtracts these quantities and produces a signal proportional to the disparity between input and output; i.e., error. Since this error may be a small quantity, it must be amplified. It is then applied to a controller which actuates the prime mover. The prime mover drives the load through the gear train so as to null the error.

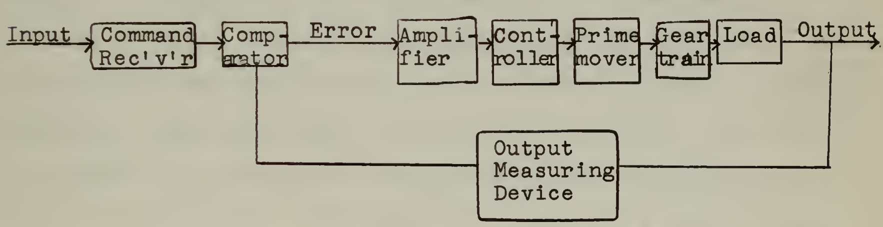


Figure II-1

A servomechanism can be considered to consist of paths through which information flows. Those paths which conduct information in the direction from input to output are termed forward paths. Those which conduct in the opposite direction are termed reverse paths. In essence, the forward paths of a

servo transform the input into the output. The reverse, or feedback, paths insure the proper degree of correspondence between input and output.

In a relatively simple system, there is usually one forward path. However, other forward paths may result from compensation considerations. To avoid ambiguity, the forward path which performs the primary function of changing input to output will be designated the main path. Other forward paths will be designated feed-forward paths.

3. Some Considerations Governing Choice of Components

The need for a servo implies that the element which generates the input has insufficient power to control the load. Since the purpose of the servo is to supply the needed power, choice of the prime mover is a logical starting point in selecting main path components. The characteristics of the load will be known. These characteristics may be expressed in terms of its inertia, the viscous friction to which it will be subjected, external torques which may act on it and the like. The specifications will also state the velocities and accelerations required of the load. This information will set a lower limit on the power and torque capacity required of the prime mover. The power supply available will further limit the choice. Other requirements on the prime mover will arise from cost, weight and size limitations, life expectancy, etc. With these limits in mind, the designer must then select a prime mover from those commercially available or else design his own.

Choice of the prime mover will normally place limits within

which a gear ratio must be chosen. Ahrendt[1] discusses the various factors which influence this choice. Such factors as low cost or smoothness of output at low velocities may place conflicting demands on the choice of gear ratio. Also, the inpedance matching aspect of gear ratio selection may be of great importance. In order to choose the optimum gear ratio, the designer must have a thorough-going knowledge of the specifications as well as of the characteristics of the prime mover.

Choice of the prime mover will probably narrowly limit choice of the controller. If a variety of controllers is available, the considerations of fineness of control versus cost, weight, etc. must be brought to bear.

Insufficient information will be available in the specifications to allow choice of the amplifier to be used. The gain to be provided will be determined by compensation considerations. This will not be known until later in the design process. When it has been determined, the amplifier may be chosen. In addition to the conditions imposed on amplifier selection by such factors as available power supply, noise generation, and operating environment, the designer must bear in mind the importance of linearity. Although sufficient band width is not difficult to obtain, saturation effects may present a problem. It is uneconomical for a system's amplifier to saturate before its prime mover does, as this means unusable power has been purchased.

The choice of comparator, measuring device and command receiver will be a question of obtaining the best linearity and accuracy within the limits set by cost, weight and size specifications. These devices are described in detail in several

servomechanism texts.[1,2,3]

4. Determination of Component Transfer Functions

Having chosen the main path components, their transfer functions must then be determined. For components which have their output independent of the frequency of the input, this merely involves determining their sensitivity. It is desirable that this sensitivity will be constant over the range of interest in the problem. If sensitivity is not constant, it must be assumed so in the remainder of the problem by assigning it a representative value.

The transfer functions of frequency dependent components may be determined mathematically by application of basic a-c circuit theory and the Laplace Transformation. This technique is described in detail by Thaler[4]. If the electrical parameters of the component are unknown, it may be necessary to conduct tests on the equipment to determine its transfer function. These may take the form of frequency or transient response tests. The results of these tests will permit derivation of the transfer function. Examples of this method are presented in references 5 and 6.

Having obtained the transfer functions of the equipment to be used, it is possible to express the system in this stage of its design by a signal flow chart or standard block diagram. This phase of design is discussed in the next chapter.

CHAPTER 3

FORMATION OF THE CHARACTERISTIC EQUATION

3.1 Introduction. The characteristic equation is an equation in s domain which represents the integro-differential equations in time space which describe the flow of information through a system.[7] The roots of this equation allow the determination of the transient performance and the stability of the physical system so described. The purpose of this chapter is to develop a standard method of forming this characteristic equation for a multiple-loop servo system.

3.2 Standard Block Diagram. A set of transfer functions are developed, as described in Chapter 2, for elements of the servomechanism. The block diagram portrays the interconnection of these various elements to form the overall system. The very nature of the block diagram assumes that the transfer function of the tandem conbination of two blocks is the product of the individual transfer functions.[8] The overall performance of the system is found by combining these blocks according to the standard rules for their manipulation.[9,10] Unfortunately for the designer, the combining of the blocks destroys their individuality and the identity of the mechanism or circuit they represent. What is needed is an alternate representation of the system which pictures it in more detail than the reduced (combined) block diagram. This could take

a new form, as in the signal-flow diagrams originated by S. J. Mason,[11] or it could be a more ordered manipulation of the block diagram, as proposed by Yaohan Chu.[10] It is to the latter method that we now devote our attention.

Certain advantages accrue from standardizing the block diagram. In this regard the following symbols are defined:

a. The Nodal Point or Node: These are summation points in the main path of the system. Each node may receive a number of signals, but delivers only one. The nodes will be lettered in sequence, a, b, c, . . n, from left (input) to right (output).

b. The Pick-off Point: This is a point at which there is one incoming signal and two or more outgoing signals. The outgoing signals are identical with the incoming one.

c. The Block Transfer Function: The transfer functions in the blocks will be designated by double subscript notation. This notation is defined for the three types of blocks as follows:

1) Direct Path Block: These blocks are noted G_{nN} where the "n" refers to the node preceding the block. "N" is the upper case form of the same letter.

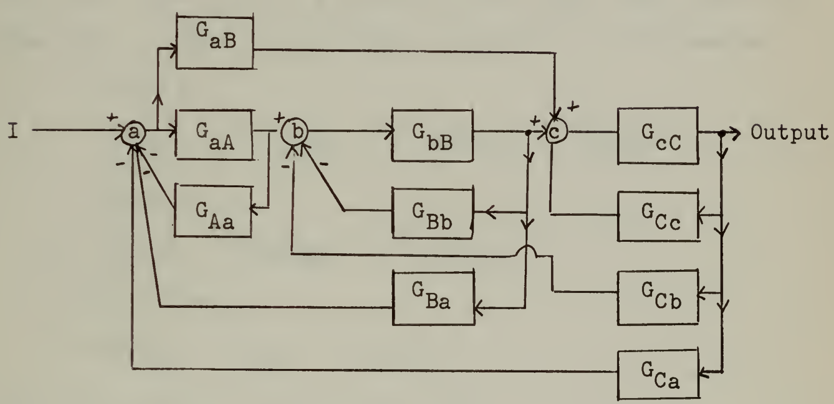
2) Feedback Block: The feedback blocks are subscripted G_{Nm}. The "N" is the same as the upper case subscript of the direct path block from which they receive their input. The "m" is the designation of the node to which they feed.

3) Feed-forward Block: Such blocks contain the symbol G_{mN}. The lower case subscript refers to the main path node

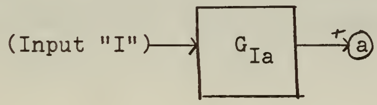
preceding the block. The upper case subscript is the same as the upper case subscript on the main path block which feeds into the same node.

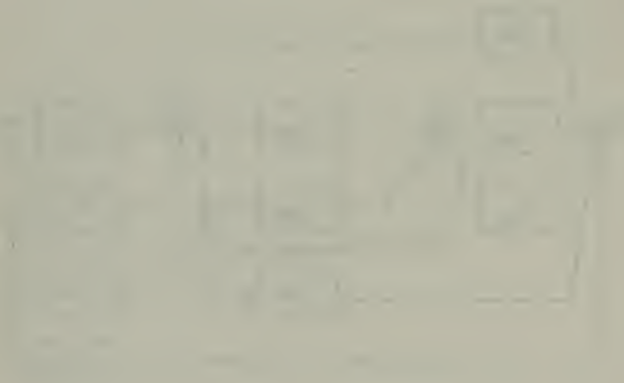
As examples of the above, consider the following three node system:

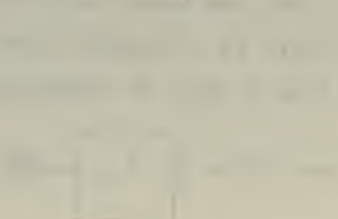
Figure 3-1



a, b, and c are nodes.  symbolizes a pickoff point. Although the system above is represented as having but one input and one output, there is no limit to the number of either. Internal noise generated in one block may be considered as an external input to the node following that block. If there is more than one input, each is designated by a Roman numeral. If there is a transfer function between an input and its entering node it will be subscripted as follows:







3.3 Mathematical Analysis Based on the Standard Block Diagram.

3.3.1 Formation of the Characteristic Determinant: We can obtain a set of simultaneous equations for the system represented by a standard block diagram by applying the network analysis technique of Guillemin.[15] This entails writing the equation for the signal summation at each node. If we identify the signal out of a node with the lower case letter of the node, we have the following equations in our example:

At Node a:

$$a = I(G_{Ia}) - c(G_{cC}G_{Ca}) - a(G_{aA}G_{Aa}) - b(G_{bB}G_{Ba})$$

At Node b:

$$b = a(G_{aA}) - b(G_{bB}G_{Bb}) - c(G_{cC}G_{Cb})$$

And, at Node c:

$$c = a(G_{aB}) + b(G_{bB}) - c(G_{cC}G_{Cc})$$

Rearranging these equations we have:

$$(1 + G_{aA}G_{Aa})a + (G_{bB}G_{Ba})b + (G_{cC}G_{Ca})c = (G_{Ia})I$$

$$(-G_{aA})a + (1 + G_{bB}G_{Bb})b + (G_{cC}G_{Cb})c = 0$$

$$(-G_{aB})a + (-G_{bB})b + (1 + G_{cC}G_{Cc})c = 0$$

Inspection of these equations reveals a symmetry of arrangement which would permit writing the equations almost without reference to the block diagram. All the forcing functions are on the right. On the left are terms which contain components and signals within the system. We can express these equations in matrix notation as follows:

$$\begin{bmatrix} 1 + G_{aA}G_{Aa} & G_{bB}G_{Ba} & G_{cC}G_{Ca} \\ -G_{aA} & 1 + G_{bB}G_{Bb} & G_{cC}G_{Cc} \\ -G_{aB} & -G_{bB} & 1 + G_{cC}G_{Cc} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} I(G_{Ia}) \\ 0 \\ 0 \end{bmatrix}$$

We call the determinant of the matrix of the coefficients the characteristic determinant of the system. To this determinant we assign the symbol $\triangle$.

3.3.2 Manipulations of the Characteristic Determinant. Independent of its physical significance, the determinant of the coefficients of a set of linear equations has certain properties. These properties are illustrated in the technique of equation solving with determinants and in the rules for expansion of a determinant.

The solution of the equations is accomplished as follows:

1) Replace the column corresponding to the desired unknown with the column of forcing functions. The resulting determinant, divided by $\triangle$, is the value of that unknown in response to these inputs.

2) Since the determinant represents a linear system, the principle of superposition is valid. Hence, in the case of multiple inputs, the output resulting from each may be computed separately and the individual results added.

3) The output obtained by replacing the last column of the determinant with the input column is the signal from the last node. It must be multiplied by the performance function following the last node to obtain the system output.

The characteristic (homogeneous) equation describing the system is obtained by setting the characteristic determinant equal to zero.

The normal rules for expanding a determinant apply to the characteristic determinant. In general they require the expansion by co-factors of some row or column. Since we will usually be interested in the final output, we should expand by elements of the last column. To the co-factor of the ith row and jth column we assign the symbol Δ_{ij}. Similarly the element in the ith row and jth column is called G_{ij}. Using this notation, the expansion of an "n by n" characteristic determinant by elements of the last (nth) column is:

$$\Delta = \sum_{i=1}^{n} G_{in}\Delta_{in}$$

As an example of the procedure described above for finding system output in response to a given input consider the following:

Input to node "a" is I, $G_{Ia} = 1$

$$\text{Output} = \frac{(G_{cC})I \begin{vmatrix} -G_{aA} & 1 + G_{bB}G_{Bb} \\ -G_{aB} & -G_{bB} \end{vmatrix}}{\Delta}$$

$$\text{Output} = \frac{I(G_{cC})[G_{aA}G_{bB} + G_{aB} + G_{aA}G_{bB}G_{Bb}]}{\Delta}$$

3.3.3 <u>Properties of the Characteristic Determinant</u>. In addition to properties inherent in all determinants the characteristic determinant has features peculiar to the system it

represents.

1) The characteristic equation, obtained by equating $\triangle$ to zero, is the sum of unity plus a number of closed loop performance functions. By the nature of the elements which comprise the determinant, these functions will be ratios of polynomials in s. When this sum is placed over a common denominator, its numerator will be the usable form of the characteristic equation of the system.

2) Each element of the characteristic determinant corresponds to a path in the standard block diagram.

3) The diagonal elements of the characteristic determinant correspond to the paths of inner loops of the nodes. The elements above these diagonal elements are all feedback paths. Those below the diagonal are feed-forward or direct paths.

4) Each feedback and feed-forward performance function appears in only one element of the determinant. Each direct path performance function may appear in more than one element.

We now have at our disposal a standard block diagram. This block diagram maintains the integrity of each element comprising the system it represents. We form this block diagram to aid us in compensating the system. To this end we need only include those nodes or pick-offs which represent points in the system physically usable for compensation. All performance functions between such points should be combined into one block to simplify the ensuing algebra. The expansion of the standard determinant representing the block diagram gives the characteristic equation of the system. Chapter 4 will relate performance specifications to this characteristic equation.

CHAPTER 4

THE EFFECT OF PERFORMANCE SPECIFICATIONS ON THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION

1. Introduction

The performance specifications state the desired system output in response to actuating or disturbing inputs. The output is specified in terms of its steady state value and its transient behavior.

The characteristic equation which was evolved in the previous chapter describes the system performance in the s domain. The standard analysis approach is to find the roots of this characteristic equation and transform it to the time domain where actual and specified performance can be compared.

The purpose of this chapter is to relate the specifications directly to the coefficients of the characteristic equation. This eliminates the need for finding the roots, plotting the response, and other laborious analysis techniques.

2. Static Performance Specifications

The static performance specifications describe the steady state value of the system output. In response to an actuating input the specification will normally state an allowable error. In response to a step we would expect no error. A following error will be stated for the response to a ramp input. A maximum allowable output is usually stated for the response to a disturbing input. These disturbances might be noise or some form of load torque.

Evaluation of the steady state output to a step, ramp or parabola of input position is most useful in studying the effect of disturbance signals. As established in Chu's dissertation[10] the performance function relating the input at any node with the output at any other node is

$$\frac{\text{Output}}{\text{Input}} = \frac{G_{jJ}\Delta_{ij}}{\Delta} \tag{4.1}$$

where: $i \triangleq$ node into which input is fed,

$j \triangleq$ node just preceding output block,

$G_{jJ} \triangleq$ performance function of output block,

$\Delta_{ij} \triangleq$ cofactor of the i^{th} row and the j^{th} column of the standard determinant.

The output in response to a given input, I(s) is

$$\frac{I(s)\ G_{jJ}\Delta_{ij}}{\Delta} \tag{4.2}$$

The steady state value of this output is obtained by applying the final value theorem as follows:

$$\text{Output}_{ss} = \underset{s \to 0}{\text{Limit}}\ s\left[\frac{I(s)\ G_{jJ}\Delta_{ij}}{\Delta}\right] \tag{4.3}$$

$$\text{Since}\quad \Delta = \frac{\text{Characteristic Equation}}{CD_{\Delta}} \tag{4.4}$$

where $CD_{\Delta} \triangleq$ common denominator of the ratios of polynomials making up the expansion of Δ.

$$\text{Output}_{ss} = \underset{s \to 0}{\text{Limit}}\left[\frac{s\ I(s)\ G_{jJ}\Delta_{ij}\ CD_{\Delta}}{\text{Characteristic Eqn.}}\right] \tag{4.5}$$

For a step of disturbance, i.e. $I(s) = A/s$

$$\text{Output}_{ss} = \lim_{s \to 0} \left[\frac{A}{a_o} \, G_{jJ} \Delta_{ij} \, CD_{\Delta} \right] \text{ since } a_o$$

is the limit of the characteristic equation as s goes to zero.

For a specified maximum output over input ratio, k, where $k = \dfrac{\text{output}_{ss}}{A}$

$$k > \lim_{s \to 0} \frac{\left[G_{jJ} \Delta_{ij} \, CD_{\Delta} \right]}{a_o} \qquad (4.6)$$

A close examination of the equation above will reveal the effects of such a specification on the a_o coefficient of the characteristic equation.

<u>Case I</u> - If k is a number not equal to zero, then a_o is specified as being greater than the $\lim_{s \to 0} \left[\dfrac{G_{jJ} \Delta_{ij} \, CD_{\Delta}}{k} \right]$.

This may or may not limit the size of a_o since this limit may be identically zero.

<u>Case II</u> - If k is specified as zero then the $\lim_{s \to 0} \left[G_{jJ} \Delta_{ij} \, CD_{\Delta} \right]$ must be zero since a_o cannot be infinite.

In Case I where the $\lim_{s \to 0} \left[G_{jJ} \Delta_{ij} \, CD_{\Delta} \right]$ is a constant we have set one condition on a_o which may be expressed in equation form as:

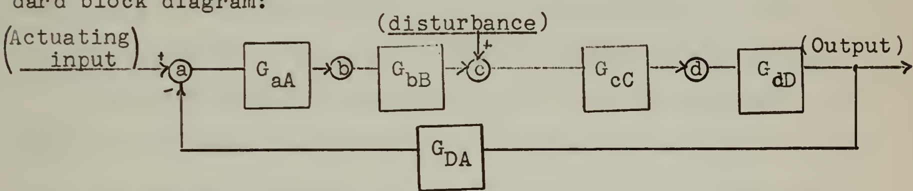
$$a_o \geq N$$

Note that the $\lim_{s \to 0} \left[G_{jJ} \Delta_{ij} \, CD_{\Delta} \right]$ may be constant containing variable gains of the uncompensated system. The condition on a_o may be met by setting these gains which determine the stiffness of the system. This will then place a lower limit on

these gains and restrict their use in future compensation.

In Case II, if the $\lim_{s \to 0} \left[G_{jJ} \Delta_{ij} \; CD_{\Delta} \right] \neq 0$, then an s must be introduced into the numerator of Δ_{ij} which does not cancel an s in its denominator; or an s must be placed in CD_{Δ} which is not also placed in the denominator of Δ_{ij}. This will raise the order of the characteristic equation. The former may be accomplished by introducing derivative feedback from any block before the output block, to the actuating input's node. The introduction of the excess s in CD_{Δ} is brought about by putting integration feedback from other than the output block to the input node to which the disturbing inputs are applied.

To illustrate the reasoning behind the statements above consider a four node system represented by the following standard block diagram:



The standard determinant for this system is:

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & G_{dD}G_{Da} \\ -G_{aA} & 1 & 0 & 0 \\ 0 & -G_{bB} & 1 & 0 \\ 0 & 0 & -G_{cC} & 1 \end{vmatrix}$$

For a disturbing input at node "c" we wish the output to be zero; but have postulated that $\lim_{s \to 0} \left[G_{dD} \, \Delta_{34} \; CD_{\Delta} \right] \neq 0$.

We need to introduce an s into one of the terms of this expression which will drive the response to zero without affecting the response to the actuating input.

The steady state value of the response to an actuating step input, A/s, is (by equation 4.5):

$$\lim_{s \to 0} \frac{A}{a_o} \left[\frac{G_{dD} \Delta_{14}}{CD_\Delta} \right] = A$$

Since G_{dD} and CD_Δ are common to both expressions a change in either will affect both responses. If the s is introduced in CD_Δ it must also appear in the denominator of Δ_{14} so that its effect is not felt in the case of the actuating input. It must not, however, appear in the denominator of Δ_{34}. The 2nd and 4th rows of Δ are contained in both cofactors so they cannot be used. The 3rd row appears in Δ_{14} but not in Δ_{34}. The only locations usable for an integration feedback which would have the desired result are positions in the 3rd row. In our example the only such feedback position is around G_{cC}.

The other method of introducing an s in the expression for steady state output in response to a disturbance, without altering the desired actuated response, is to put an s in the numerator of Δ_{34} which does not appear in the numerator of Δ_{14}. This is accomplished by inserting derivative feedback in a first row position. The 1st row is the only one in Δ_{34} which is not also in Δ_{14}.

In higher order systems there may be peculiarities of the determinant which would allow other positions to be used, or which would eliminate some of the above positions. Those explained above are the only positions possible if all terms

in the determinant are present.

Computation of the output quantity is of prime importance in devising methods of making a system insensitive to disturbing inputs. In response to an actuating input we might better express steady state performance in terms of error since this permits an immediate and direct check on the accuracy of the system. If error is defined as the difference between the actuating input and its desired output, then the error equation is:

$$E = \text{Input} - \text{Output} \tag{4.7}$$

This may be rewritten as:

$$E = \left[1 - \frac{\text{Output}}{\text{Input}}\right] \text{Input.} \tag{4.8}$$

Expressing Output/Input as per equation(4.1) we have:

$$E = \left[\frac{1 - G_{jJ}\Delta_{ij}}{\Delta}\right] I(s) \tag{4.9}$$

By the final value theorem, and again expressing Δ as in equation (4.4), we have

$$\text{Error}_{ss} = \lim_{s \to 0} sI(s) \left[\frac{\text{Char.Eqn.} - G_{jJ}\Delta_{ij}\ CD_{\Delta}}{\text{Char. Eqn.}}\right] \tag{4.10}$$

In response to I(s) = A/s:

$$\text{Error}_{ss} = \lim_{s \to 0} A \left[\frac{\text{Char Eqn} - G_{jJ}\Delta_{ij}\ CD_{\Delta}}{\text{Char. Eqn.}}\right] \tag{4.11}$$

Since the steady state error in response to a step will be defined as zero we can write:

$$0 = \frac{\underset{s \to 0}{\text{Limit}} \; A \left[\text{Char.Eqn.} - G_{jJ}\Delta_{ij} \; CD_{\Delta} \right]}{\underset{s \to 0}{\text{Limit}} \quad \text{Char. Eqn.}} \tag{4.12}$$

Since $\underset{s \to 0}{\text{Limit}} \left[\text{Characteristic Equation} \right] = a_o \neq 0$, then:

$$a_o = \underset{s \to 0}{\text{Limit}} \left[G_{jJ}\Delta_{ij} \; CD_{\Delta} \right] \tag{4.13}$$

This defines another condition on a_o, but one which will automatically be met if there is unity feedback from the output block to the actuating input node.

In response to an input ramp, i.e. $I(s) = B/s^2$:

$$\text{Error}_{ss} = \underset{s \to 0}{\text{Limit}} \; \frac{B}{s} \left[\frac{\text{Char. Eqn.} - G_{jJ}\Delta_{ij} \; CD_{\Delta}}{\text{Char. Eqn.}} \right] \tag{4.14}$$

For zero error in response to a step, the $\underset{s \to 0}{\text{Limit}}$ of the Characteristic Equation equals the $\underset{s \to 0}{\text{Limit}} \; G_{jJ}\Delta_{ij} \; CD_{\Delta} = a_o$. Hence the constant parts of the numerator of equation (4.14) are equal. This means that s is a factor of the numerator.

$$\text{Error}_{ss} = \underset{s \to 0}{\text{Limit}} \; \frac{B}{a_o} \left[\frac{a_1 s + a_o - G_{jJ}\Delta_{ij} \; CD_{\Delta}}{s} \right] \tag{4.15}$$

$$\frac{\text{Error}}{B}_{ss} = \frac{a_1}{a_o}$$

This equation puts a limit on the ratio of a_1/a_o in the compensated characteristic equation.

This ratio may not be compatible with the a_1/a_o ratios of any of the model equations formed from dynamic specifications. The restriction on a_1/a_o may be relieved by making

the ramp following error identically zero, i.e., raise system to Type II.

All the conditions on equation order, compensator location and coefficient magnitude which arise from consideration of static specifications should be held for simultaneous solution with similar conditions yielded by the dynamics of the problem.

C. Dynamic Performance Specifications

The dynamic performance specifications describe the desired transient performance of the system. Since they are meaningless in an unstable system, they all imply stability even if they do not state it.

As opposed to the static specifications, where there is some uniformity, there are nearly as many ways of expressing a desired transient behavior as there are Process Engineers. The following summary[12], while not intended to be all inclusive, lists the most common specifications and discusses how to calculate them.

THE TEN MOST COMMON DYNAMIC PERFORMANCE SPECIFICATIONS

NAME	TYPE	METHOD OF COMPUTATION
1 Gain Margin	Stability specification	Compute maximum gain for stability by Routh-Hurwitz criterion. Ratio of maximum stable gain to actual gain is gain margin.

2 Phase Margin	Frequency domain stability specification.	On Nyquist diagram, the angle between curve and negative real axis at the unit circle. On Bode plot, find ω_c (frequency of unity gain), compute phase shift at ω_c, and subtract from 180 deg. For a well behaved system, a phase margin of 45 deg. means one overshoot in response to a step.
3 M Peak	Frequency domain stability specification.	The M curve is the magnitude of the closed-loop response vs. frequency. It can be calculated from open-loop transfer function or from values picked off Nyquist plot. M peak is defined as the peak of closed-loop frequency response normalized to low frequency value. $M = \frac{A}{1 - A}$ Circles of constant M can be drawn on Nyquist

		plot. The locus of the system must remain outside given M peak circle. (An M peak of 1.4 usually means one overshoot)
4 Damping Ratio, ζ	Stability specification.	Can be calculated from closed-loop transfer function or found directly from root-locus plot. Damping ratio is defined as ζ in a quadratic term of the form: $s^2 + 2\zeta\omega_n s + \omega_n^2$ On the root-locus diagram, ζ equals cos θ where θ is the angle between the horizontal and the vector directed toward the gain point on the locus.
5 Damping Factor, $\zeta\omega_n$ or Decrement Factor	Stability Specification	May be found directly from the root-locus. Damping factor is defined by the factors of a quadratic system, as was damping ratio; it is the real portion of the roots.

		Where the imaginary portion of the root determines the frequency of the damped sinusoidal response of a system, the real portion determines the rate of decay.
6 Percent Overshoot	Direct evaluation of system stability	May be calculated from response of system to step function input. Defined as peak of the response to a step-function input as a percentage of final value.
7 Bandwidth	Frequency domain speed-of-response specification	Usually defined as the frequency at which the closed-loop frequency response falls to $1/\sqrt{2}$ its low (zero) frequency value. Bandwidth is available directly from the M curve of the system. An analogous definition would be the crossover frequency of the system.

8 Rise Time	Speed-of-response specification	One of the simplest definitions of rise time is 1/bandwidth, another is $1/\omega_{M\ peak}$. Rise time is also defined as time to first zero of error.
9 "Settle-Out" Time or "Synchronizing" Time	Overall performance Specification	Settle-out time can be calculated analytically by the inverse Laplace transform, or by picking values off the root locus plot. This specification is most convenient if an analog computer is available. Settle-out time is defined as the time required for system response to a step-function input to approach and remain within a given tolerance of the final value. It is usually necessary to specify maximum overshoot and steady-state error in addition to settle-out time.

10		
ITAE or Integrated Value of the Product of Time and Absolute Value	Overall Performance Specification	The ITAE specification is defined by the integral $\int_0^\infty t\lvert e\rvert dt$ Minimize the value of this integral for optimum system performance. This specification yields a number that depends on system parameters. It would probably be an improvement to generalize it by normalizing to the rated output quantity. The dimensions would then be sec^2. Another possible generalization would be to integrate to some arbitrary large number, such as 10 time constants, rather than to infinity. This would permit the specification to be applied to responses for which the system yields a finite final error. ITAE is one of several

attempts to define an overall figure of merit for system operation. Rather than simply summing the total error of a system, the error is progressively weighted more heavily as time goes on. This puts a premium on rapid, accurate settle-out, and allows for unavoidable initial large error.

Due to the complexity and variety of responses of systems of order higher than second, very little information is available in the literature concerning the characteristics of their response. However, the response of a second order system is easily computed, fairly simple, and can be characterized by two parameters. If the system has an oscillatory response, the parameters most commonly used are ζ and ω_n, the damping ratio and undamped natural frequency respectively. The use of these two parameters is quite natural since the coefficients of the system quadratic equation are simple functions of ζ and ω_n.

As seen above, dynamic specifications will not always be given in terms of ζ and ω_n, but translation to this form

is possible. As an example, Instrument Engineering, Vol. II[13] provides graphs of calculated second order response in both the time and frequency domains which may be used for this purpose. Chu[14] gives several rules for translating time domain specifications into ζ and ω_n and Thaler gives others.[9] Thaler also includes simplified graphs giving relationships between frequency domain specifications and ζ and ω_n as well as rules of thumb relating phase and gain margin to height of resonant peak.

Translation of specifications into ζ and ω_n or relatable parameters may be other than precise. Analytical relationships may not exist, as in the case of phase and gain margins. Specifications may put varying demands on the physical components of the system such that time dependent coefficients are required. They may prescribe a performance unrealizable by an order lower than third. In cases such as these, reversion to trial-and-error design techniques may be dictated. If the response of a higher order system can be made to closely approximate a second order system, then adequate correlation between response and the ζ, ω_n type specifications is possible. It is to the problem of assuring dominance of a given quadratic factor that we now turn our attention.

A quadratic factor is said to have "dominance" if the response to the higher order system closely approximates that of the specified quadratic. A high "degree of dominance" means a close approximation, i.e. small deviation of actual from second order response. The use for which the system is intended and the strictness of the specifications will

indicate the degree of dominance required.

In an oscillatory second order system, the correlation between coefficients of the characteristic equation and the ζ and ω_n specifications is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \tag{4.16}$$

If our system has a cubic characteristic equation with one pair of oscillatory roots it will appear as follows on the s-plane:

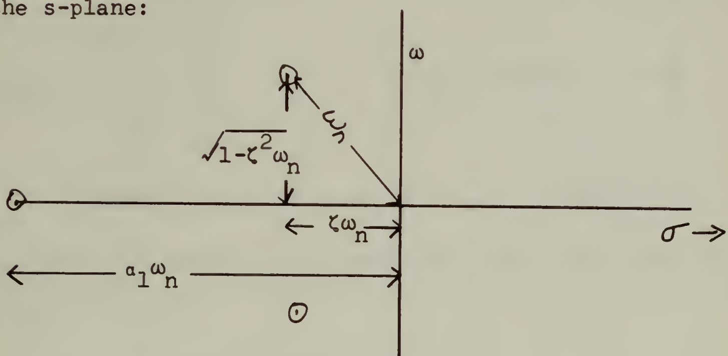


Figure 4-1

where α_1, the dominance factor, is defined as shown in figure 4-1. The model characteristic equation in terms of α_1, ζ and ω_n is:

$$(s+\alpha_1\omega_n)(s^2+2\zeta\omega_n s+\omega_n^2) = 0$$

$$s^3+(2\zeta+\alpha_1)\omega_n s^2+(2\zeta\alpha_1+1)\omega_n^2 s+a_1\omega_n^3 = 0 \tag{4.17}$$

The degree to which the response of this system approximates a second order response depends on the size of α_1. A detailed examination of the effect of α_1 on the deviation of a third

order response from that of a second order is shown in Appendix A.

A system with a fourth order characteristic equation might take either of the following s-plane configurations:

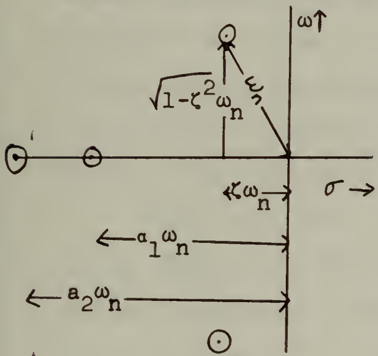


Figure 4-2a

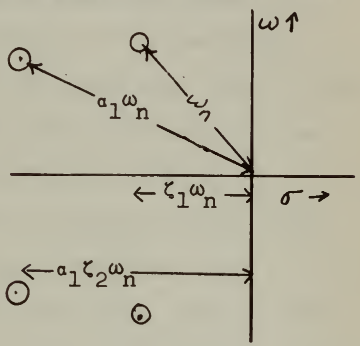


Figure 4-2b

The model characteristic equation in the case shown in figure 4-2a is:

$$(s+\alpha_1\omega_n)(s+\alpha_2\omega_n)(s^2+2\zeta\omega_n s+\omega_n^2) = 0$$

$$s^4+(2\zeta+\beta_1)\omega_n s^3+(1+2\zeta\beta_1+\beta_2)\omega_n^2 s^2+(\beta_1+2\zeta\beta_2)\omega_n^3 s+\beta_2\omega_n^4 = 0 \tag{4.18}$$

For the case in figure 4-2b the model characteristic equation is:

$$(s^2+2\alpha_1\zeta_2\omega_n s+\alpha_1^2\omega_n^2)(s^2+2\zeta_1\omega_n s+\omega_n^2) = 0$$

$$s^4+(2\zeta_1+2\alpha_1\zeta_2)\omega_n s^3+(1+4\alpha_1\zeta_1\zeta_2+\alpha_1^2)\omega_n^2 s^2+(2\alpha_1\zeta_2+2\zeta_1\alpha_1^2)\omega_n^3 s$$

$$+\alpha_1^2\omega_n^4 = 0 \tag{4.19}$$

In choosing the α's for the case in figure 4-2a a reference to the curves of Appendix A is helpful but not exact. Each α must be sufficiently large so that its effect alone would be within the tolerances allowed. A conservative choice of the individual α's is necessary to insure against the possibility of accumulating error of the same sign. Determination of the effect of more than one real root is a fruitful area for further work.

In the case where there are two pairs of oscillatory roots, there is a slightly different problem. Here it is not only the decay of the non-dominant pair, but the size of the coefficient of their sinusoidal contribution to the time response, which determines their effect. In the extreme case, where they are on the imaginary axis, this coefficient is approximately $1/\alpha^2$ times that of the dominant pair.

The extreme condition described above also serves to place a lower limit on the size of the coefficients of the characteristic equation in order to have stability. If $\zeta_2 = 0$ the minimum value of the coefficients of the fourth order characteristic equation are defined as follows:

$$s^4+(2\zeta_1)\omega_n s^3+(1+\alpha_1^2)\omega_n^2 s^2+(2\zeta_1\alpha_1^2)\omega_n^3 s+\alpha_1^2\omega_n^4 = 0 \quad (4.20)$$

Similar examination of higher order characteristic equations shows the relationship between α's, ζ and ω_n, and their coefficients. Appendix B lists the various forms of these coefficients, including minimum stability values, for model equations up to seventh order.

As a result of the considerations of static and dynamic specifications we can now write model characteristic equations for a system which will exhibit a specified response. Considerations of chapters II and III have given us an actual characteristic equation which must be forced into this specified configuration by compensation.

CHAPTER 5

EFFECT OF COMPENSATION ON THE COEFFICIENTS OF THE CHARACTERISTIC EQUATION

5.1 Introduction. In Chapter 3 we formed the characteristic equation which represents the components in our servomechanism. The static and dynamic specifications were related to coefficients of model characteristic equations, and these are tabulated in Appendix B. The third phase of our compensation procedure consists of determining the effect of compensation on the coefficients of the characteristic equation. This effect can be considered under two headings:

a) The Effect of Compensator Location

b) The Effect of Compensator Type.

The purpose of this chapter is to show how to quickly determine each of these effects. The relationship between them and the coefficients will be examined. A notation will be introduced which permits rapid evaluation of the effect of all possible compensators, without repeated expansions of the characteristic determinant.

The separation of the two effects is possible largely because of the unique manner in which the addition of a compensator alters the characteristic determinant. Before considering the two effects separately we will examine the effect of compensation on the determinant of a general, four-node

uncompensated servomechanism.

5.2 <u>The Effect of Compensation on the Characteristic Determinant</u>. In studying the effect of a compensator on the characteristic determinant, we will use the four node system in figure 5.1.

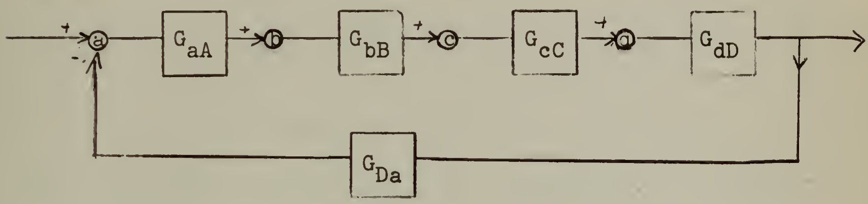


Figure 5-1

The characteristic determinant for this system is:

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & G_{dD}G_{Da} \\ -G_{aA} & 1 & 0 & 0 \\ 0 & -G_{bB} & 1 & 0 \\ 0 & 0 & -G_{cC} & 1 \end{vmatrix}$$

The characteristic determinant is single valued, and so its value is independent of the method used in expanding it.

For example: When expanded by cofactors of the fourth column

$$\Delta = (-1)^5(-G_{dD}G_{Da}) \begin{vmatrix} -G_{aA} & 1 & 0 \\ 0 & -G_{bB} & 1 \\ 0 & 0 & -G_{cC} \end{vmatrix} + (-1)^8(1) \begin{vmatrix} 1 & 0 & 0 \\ -G_{aA} & 1 & 0 \\ 0 & -G_{bB} & 1 \end{vmatrix}$$

$$\triangle = G_{dD}G_{Da}G_{aA}G_{bB}G_{cC}+1$$

A similar expansion by cofactors of the third row yields:

$$\triangle = (-1)^5(-G_{bB})\begin{vmatrix} 1 & 0 & G_{dD}G_{Da} \\ -G_{aA} & 0 & 0 \\ 0 & -G_{cC} & 1 \end{vmatrix} + (-1)^6(1)\begin{vmatrix} 1 & 0 & G_{dD}G_{Da} \\ -G_{aA} & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\triangle = G_{bB}G_{dD}G_{Da}G_{cC}G_{aA} + 1$$

Note that both expansions give the same result, and indeed, any expansion would give this result.

If we were to add a compensator to any path previously open, we would put a transfer function in a determinant position previously zero. As an example, consider closing the path from the output of block G_{bB} to node "a" with an element whose transfer function is G_{Ba}. The characteristic determinant of this modified system is:

$$\triangle_m = \begin{vmatrix} 1 & G_{bB}G_{Ba} & 0 & G_{dD}G_{Da} \\ -G_{aA} & 1 & 0 & 0 \\ 0 & -G_{bB} & 1 & 0 \\ 0 & 0 & -G_{cC} & 1 \end{vmatrix}$$

Expanding by cofactors of the first row:

$$\triangle_m = (-1)^2(1)\begin{vmatrix} 1 & 0 & 0 \\ -G_{bB} & 1 & 0 \\ 0 & -G_{cC} & 1 \end{vmatrix} +(-1)^3(G_{bB}G_{Ba})\begin{vmatrix} -G_{aA} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -G_{cC} & 1 \end{vmatrix} +..$$

$$(G_{dD}G_{Da})\begin{vmatrix} -G_{aA} & 1 & 0 \\ 0 & -G_{bB} & 1 \\ 0 & 0 & -G_{cC} \end{vmatrix}$$

$$\Delta_m = 1 + G_{bB}G_{Ba}G_{aA} + G_{dD}G_{Da}G_{aA}G_{bB}G_{cC}$$

This is merely the old Δ with the term $G_{bB}G_{Ba}G_{aA}$ added.

A similar effect is observed is an existing path is paralleled. Suppose an element whose performance function is G'_{aA} was put in parallel with G_{aA}. This would make the transfer function between nodes "a" and "b" equal $(G_{aA} + G'_{aA})$. Since no new position in the determinant was filled, the expansion remains:

$$\Delta_m = (G_{aA} + G'_{aA})G_{bB}G_{cC}G_{dD}G_{Da} + 1$$

$$\Delta_m = G_{aA}G_{bB}G_{cC}G_{dD}G_{Da} + 1 + G'_{aA}G_{bB}G_{cC}G_{dD}G_{Da}$$

This again is the old Δ with a new term added. We can generalize the above observations as follows:

$$\Delta_m = \Delta + G_{ij}\Delta_{ij} \tag{5.1}$$

where G_{ij} is the element added to the i^{th} row, j^{th} column of the determinant as a result of adding the compensator. Δ_{ij} is the signed minor (cofactor) of this determinant position. It is the effect of $G_{ij}\Delta_{ij}$ on the coefficients of the characteristic equation that we must determine.

All the terms in equation (5.1) contain ratios of

polynomials in "s". These polynomials come from the products of terms G_{nN}, G_{Nm}, etc. in the characteristic determinant. These transfer functions represent components of the servo-mechanism. By their physical nature, components normally contribute a time lag to the system. This lag appears in the transfer function as an (s+a) factor in the denominator. The expansions of $\triangle$ and $\triangle_m$ consist of unity plus a series of products of such transfer functions. Because of the prevalence of lagging components these products will have numerators of order less than (or at best equal to) their denominators. When the terms comprising $\triangle$ are placed over a common denominator, the numerator so formed is the usual form of the characteristic equation. The common denominator will be the product of all the non-repeated (s+a) factors, and the highest power of s in any of the individual product's denominators. The numerator contains unity times this common denominator. It is this contribution to the numerator that contains the highest power of s, and thus sets the order of the characteristic equation.

With these thoughts in mind, consider again equation (5.1):

$$\triangle_m = \triangle + G_{ij}\triangle_{ij} \tag{5.1}$$

$$\frac{\text{Modified Char.Eqn.}}{\text{C.D.}_{\triangle m}} = \frac{\text{Char.Eqn.}}{\text{C.D.}_{\triangle}} + G_{ij}\triangle_{ij} \tag{5.2}$$

$$\text{Modified Char. Eqn.} = \left[\frac{\text{C.D.}_{\triangle m}}{\text{C.D.}_{\triangle}}\right] \text{Char. Eqn.} + G_{ij}\triangle_{ij}\text{CD}_{\triangle m} \tag{5.3}$$

From this we see two types of changes to the coefficients of

the characteristic equation:

a) If $\frac{CD_{\triangle m}}{CD_{\triangle}}$ is not unity, i.e., if the compensator added new (s+a) factors or a higher power of s to the common denominator, the old characteristic equation will be multiplied by the product of these added terms. This will raise the order of the characteristic equation.

b) The coefficients of s^k in $G_{ij}\triangle_{ij}CD_{\triangle m}$ will be added to the coefficients of s^k in $\left[\frac{CD_{\triangle m}}{CD_{\triangle}}\right]$ characteristic equation.

Had the purpose of this chapter been merely to show the effect of a compensator on the coefficients of the characteristic equation, it would be fulfilled. One would only have to apply equation (5.3) for each compensator-path combination. For an "n" node system and "m" possible compensators, this requires (n x n x m) applications of equation (5.3). This is a process that requires many, many hours of algebra. It usually just isn't done. The designer relies on his previous experience to help him pick a usable path and is likely to overlook one which would have done the job easier. The remainder of this chapter will develop a technique for examining all possible paths in a reasonable time, and for displaying this information in a useful manner.

5.3 The Effect of Compensator Location. We have seen (equation 5.1) that adding a compensator has the effect of adding a term, $G_{ij}\triangle_{ij}$, to the characteristic determinant. Factoring the performance function of the compensator from the term $G_{ij}\triangle_{ij}$ will leave the effect of closing the path

in which the compensator was inserted. It is this effect that we call the "Effect of Compensator Location". $G_L = \frac{G_{ij}}{G_C}$.

It is the term which would appear as characteristic determinant element i-j, if the path represented by that element were merely closed.

$$G_{ij} = G_L, \text{ if } G_C = 1$$

Rewriting equation (5.3) for this condition:

$$\text{Modified Char. Eqn.} = \left[\frac{CD\Delta m}{CD\Delta}\right] \text{Char. Eqn.} + G_C G_L \Delta_{ij} CD_{\Delta m} \tag{5.4}$$

Since there is no denominator in the performance function $G_C = 1$, there are no new factors added to the common denominator by the inclusion of $G_C G_L \Delta_{ij}$. $CD_{\Delta m} = CD_{\Delta}$.

$$\text{Modified Char. Eqn.} = \text{Char. Eqn.} + G_L \Delta_{ij} CD_{\Delta} \tag{5.5}$$

$$\text{Modified Char. Eqn.} - \text{Char. Eqn.} = G_L \Delta_{ij} CD_{\Delta} \tag{5.6}$$

(Modified Char. Eqn. - Char. Eqn.) will be a polynomial in s containing only those powers of s whose coefficients were changed by closing the path. If we could quickly determine the powers of s in $G_L \Delta_{ij} CD_{\Delta}$, for all possible paths, we would know what coefficients of the characteristic equation were altered by closing those paths.

To accomplish this we will introduce a new notation. This notation will describe the powers of s contained in the polynomials which comprise the terms in equation (5.6).

Since these polynomials in s arise from the multiplication or division of factors (s+a) or "s", they will contain all the successive powers of s between the highest and lowest power present. The symbol so used will be called an "Order Number", and will consist of two numbers, one Arabic and one Roman, i.e. (A,R). The highest power of s in the polynomial is s^{A+R}, and the lowest power of s is s^R. All powers of s between these are present. For example:

If (Order of P(s)) = (3,I), then $P(s) = a_4s^4+a_3s^3+a_2s^2+a_1s$.

The algebra of these Order Numbers is much like that of exponents. When two polynomials are multiplied, their Order Numbers add to give the Order Number of the product. If one polynomial in s is divided by another, the difference in their Order Numbers is the Order Number of the quotient. Order Numbers add and subtract as follows:

$$(A_1,R_1) \pm (A_2,R_2) = (A_1 \pm A_2,\ R_1 \pm R_2)$$

As an example of this use of Order Numbers, consider:

$$(s+a)(s^2+bs) = s^3 + (a+b)s^2+abs$$

Order Number of (s+a) = (1,0)

Order Number of (s^2+bs) = (1,I)

(1,0) + (1,I) = (2,I)

Order Number of $s^3 + (a+b)s^2+abs$ is (2,I)

Since the terms of equation (5.6) are polynomials in s of the type representable by Order Numbers we can rewrite this equation as follows:

$$\text{(Coef. Indicator)} = (\text{Order of } G_L) + (\text{Order of } \triangle_{ij}) + (\text{Order of } CD_{\triangle}) \qquad (5.7)$$

Before examining the use of equation (5.7), it is well to become more familiar with these Order Numbers, and to see how they are found for the terms of the equation.

The Arabic number equals the highest power of s arising from the combination of the terms of the form (s+a).

Example 1: In (s+a)(s+b)(s+c), the Arabic number is 3.

Example 2: In $\frac{(s+a)(s+b)}{(s+c)}$, the Arabic number is 1.

The Roman number equals the net power of s arising from terms of the form s^n.

Example 3: In s^3, the Roman number is III.

Example 4: In $\frac{s^m}{s^n}$, the Roman number is (m-n).

As examples of polynomials in which both Arabic and Roman numbers occur:

Example 5: The order of $\frac{(s+a)}{s(s+b)}$ is (0,I)

Example 6: The order of $s^2(s+a)(s+b)(s+c)$ is (3,II).

Example 7: The order of 0 is $(-\infty,-\infty)$. To simplify the notation, this will be designated (-,-). Note that (A,R)+ (-,-) = (-,-). Any sum containing (-,-) has no value. It is as if you multiplied by zero.

The Order Numbers comprising the left side of equation (5.7) are obtained as follows:

(Order of G_L'): In determinant positions corresponding to feedback paths, G_L is the transfer function of the block immediately preceding the feedback pickoff. It would be of the form of Example 5 above. In determinant positions corresponding to feed forward, or parallel to a direct path, it

equals unity. The Order Number of a constant is (0,0).

(Order of $\triangle_{ij}$): This could be found by expanding the minor, $\triangle_{ij}$, and then applying the rules exemplified above. A simpler method consists of replacing each existing term in the characteristic determinant with its Order Number. The minors are then expanded by the usual determinant rules with the following exceptions:

1) The Order Numbers are added where the terms they represent would have been multiplied.

2) Arabic and Roman numbers are added separately.

3) The largest Arabic and the largest Roman numbers resulting from these additions comprise the Order Number of the minor.

(Order of $CD_{\triangle}$): The common denominator of $\triangle$ will have already been found in the determination of the characteristic equation (Chap. 3). Apply the rules above to this denominator as in Example 6.

With the use of the Order Number technique, a matrix, showing the results of equation (5.7) for each path, can be formed. This "Location Matrix" is constructed as follows:

a) Replace the existing transfer functions in the Characteristic determinant by their Order numbers.

b) Evaluate the (Order of $\triangle_{ij}$) for each i-j position usable for compensation. (Normally all positions)

c) To this (Order of $\triangle_{ij}$) add (Order of G_L) for the position concerned.

d) Add (Order of $G_L\triangle_{ij}$) to (Order of $CD_{\triangle}$) and display the resulting Order Numbers in a matrix. Each Order Number

will occupy a position in the matrix corresponding to the position of its path in the Characteristic Determinant.

This matrix will indicate which coefficients can be changed by closing a given path.

To clarify the ideas presented in this section, the location matrix for a sample problem will be formed. Consider the system of Figure 5-2.

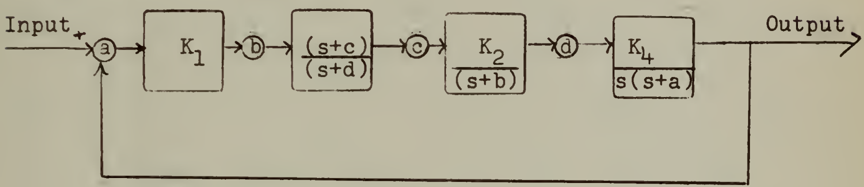


Figure 5-2

This system has the following characteristic determinant:

$$\triangle = \begin{vmatrix} 1 & 0 & 0 & \frac{K_4}{s(s+a)} \\ -K_1 & 1 & 0 & 0 \\ 0 & \frac{-(s+c)}{(s+d)} & 1 & 0 \\ 0 & 0 & \frac{-K_2}{(s+b)} & 1 \end{vmatrix}$$

Expanding this in terms of elements of the fourth column:

$$\triangle = 1 + \frac{K_1 K_2 K_4 (s+c)}{s(s+a)(s+b)(s+d)}$$

$$\triangle = \frac{s(s+a)(s+b)(s+d)+K_1 K_2 K_4 (s+c)}{s(s+a)(s+b)(s+d)}$$

Expanding the numerator and introducing standard symbols for coefficients:

$$\frac{\text{Char.Eqn.}}{CD_{\triangle}} = \frac{s^4+a_3s^3+a_2s^2+a_1s+a_o}{s(s+a)(s+b)(s+d)}$$

Applying the rules for formation of a Location Matrix:

Step a:

$$\triangle = \begin{vmatrix} (0,0) & (-,-) & (-,-) & (-1,-I) \\ (0,0) & (0,0) & (-,-) & (-,-) \\ (-,-) & (0,0) & (0,0) & (-,-) \\ (-,-) & (-,-) & (-1,0) & (0,0) \end{vmatrix}$$

Step b: $\triangle_{11}$ - (0,0) $\triangle_{12}$ - (0,0) $\triangle_{13}$ - (0,0) $\triangle_{14}$ - (-1,0)
$\triangle_{21}$ - (-2,-I) $\triangle_{22}$ - (0,0) $\triangle_{23}$ - (0,0) $\triangle_{24}$ - (-1,0)
$\triangle_{31}$ - (-2,-I) $\triangle_{32}$ - (-2,-I) $\triangle_{33}$ - (0,0) $\triangle_{34}$ - (-1,0)
$\triangle_{41}$ - (-1,-I) $\triangle_{42}$ - (-1,-I) $\triangle_{43}$ - (-1,-I) $\triangle_{44}$ - (0,0)

The Order numbers of the G_L's are as follows:

$G_{L_{11}}$ - (0,0) $G_{L_{12}}$ - (0,0) $G_{L_{13}}$ - (-1,0) $G_{L_{14}}$ - (-1,-I)

$G_{L_{21}}$ - (0,0) $G_{L_{22}}$ - (0,0) $G_{L_{23}}$ - (-1,0) $G_{L_{24}}$ - (-1,-I)

$G_{L_{31}}$ - (0,0) $G_{L_{32}}$ - (0,0) $G_{L_{33}}$ - (-1,0) $G_{L_{34}}$ - (-1,-I)

$G_{L_{41}}$ - (0,0) $G_{L_{42}}$ - (0,0) $G_{L_{43}}$ - (0,0) $G_{L_{44}}$ - (-1,-I)

Step c: (Order of $G_L\triangle_{ij}$) - position in the array denotes subscripts.

(0,0)	(0,0)	(-1,0)	(-2,-I)
(-2,-I)	(0,0)	(-1,0)	(-2,-I)
(-2,-I)	(-2,-I)	(-1,0)	(-2,-I)
(-1,-I)	(-1,-I)	(-1,-I)	(-1,-I)

(Order of $CD_{\triangle}$) equals (3,I).

Step d:

$$\text{Location Matrix} = \begin{bmatrix} (3,I) & (3,I) & (2,I) & (1,0) \\ (1,0) & (3,I) & (2,I) & (1,0) \\ (1,0) & (1,0) & (2,I) & (1,0) \\ (2,0) & (2,0) & (2,0) & (2,0) \end{bmatrix}$$

Step b, finding the Order of each minor, merely entails scanning the auxiliary determinant of step a to find the largest Arabic and the largest Roman sums present. The presence of numerous (-,-) terms in an uncompensated system makes this a few minutes work. This is in comparison to the hours of algebra required to evaluate $G_{ij}\Delta_{ij}CD_{\Delta}$ for every possible compensator.

The Location Matrix is the sum of the terms of Equation (5.7) for all possible compensation paths. The information in the Location Matrix, modified by the effect of compensator type, will indicate changeable coefficients of the Characteristic Equation.

5.3. The Effect of Compensator Type

5.3.1 Introduction

Examination of the type of compensator yields information on:

a. Which coefficients of the Characteristic Equation will be changed, and

b. the extent of that change.

It is the Order Number of the compensator in conjunction with the Location Matrix that indicates which coefficients will be changed. The performance function indicates the magnitude of the change.

Table V-1 lists the compensator types to be considered, with their Order Number and Performance Function.

Table V-1 Elementary Compensators

TYPE	ORDER NUMBER	PERFORMANCE FUNCTION
Sensitivity	(0,0)	K
Differentiator	(0,I)	s
First Order Lead Filter	(1,0)	(s+a)
Second Order Lead Filter	(2,0)	(s^2+bs+c)
Integrator	(0,-I)	1/s
First Order Lag Filter	(-1,0)	1/(s+a)
Second Order Lag Filter	(-2,0)	$1/(s^2+bs+c)$

5.3.2 Determination of Which Coefficients are Changed

The Order Number of the compensator, when added to the element of the Location Matrix corresponding to its path, gives a new Coefficient Indicator. As previously defined, the (A,R) pair resulting from this addition indicates which coefficients of the Characteristic Equation are modified; i.e., a_{A+R}, a_{A-1+R}, - - -, a_R.

There are three situations in which addition of a compensator will raise the order of the Characteristic Equation. If the order is raised, there will be two changes to the coefficients. One change will be to all coefficients due to merely raising the order. The change dictated by the Coefficient Indicator will be an additional change to the coefficients of the higher order equation.

Before examining the quantitative effects of compensation

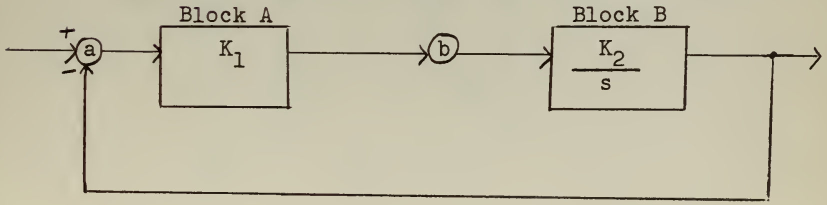
on the coefficients, the three cases where the order of the Characteristic Equation is raised will be considered:

Case I - This situation arises when integrator compensation is used. If the addition of a compensator results in one of the terms of

$$1 + \sum_r L_r + \sum_r \prod_2 L_r + \sum_r \prod_3 L_r + - - -$$

having more integrations than the overall uncompensated closed loop performance function, the order of the Characteristic Equation will be raised. In the above expression, L_r signifies loop performance function. $\prod_k$ signifies products of loop performance functions taken k at a time. That the order of the Characteristic Equation is raised in this situation is apparent if the Characteristic Equation is derived from the system performance function obtained by application of Mason's Rule.[11]

A simple example of this effect may be seen by considering the following system:



$$\text{Where } \triangle = \begin{bmatrix} 1 & \frac{K_2}{s} \\ -K_1 & 1 \end{bmatrix}$$

$$\triangle = 1 + \frac{K_1K_2}{s}$$

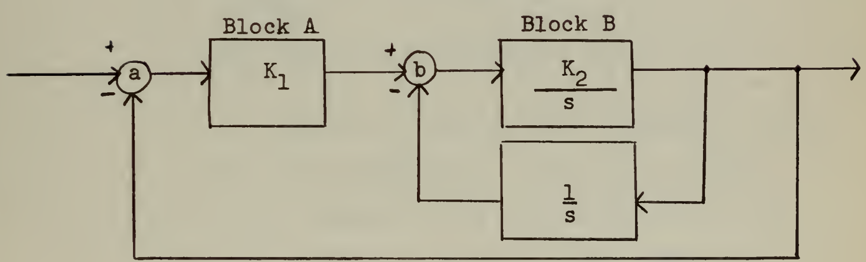
$$\frac{\text{Char. Eqn.}}{\text{CD}\triangle} = \frac{s + K_1K_2}{s}$$

$$\text{Char. Eqn.} = s + K_1K_2 = 0$$

For this system the Location Matrix is:

$$\text{L. M.} = \begin{bmatrix} (0,I) & (0,0) \\ (0,0) & (0,0) \end{bmatrix}$$

Consider integration feedback from the output of Block B to Node b:



$$\text{Here, } \triangle = \begin{bmatrix} 1 & \frac{K_2}{s} \\ -K_1 & 1 + \frac{K_2}{s^2} \end{bmatrix}$$

$$\triangle = 1 + \frac{K_2}{s^2} + \frac{K_1K_2}{s}$$

$$\frac{\text{Char. Eqn.}}{CD\triangle} = \frac{s^2 + K_1K_2 s + K_2}{s^2}$$

$$\text{Char. Eqn.} = s^2 + K_1K_2 s + K_2 = 0$$

Notice that the order of the Characteristic Equation has been raised by this combination of compensator and path. If the integration had been fed back from the output of Block A to Node a, a loop performance function higher than first order would not have been generated and the order of the equation would not have been raised. In this case,

$$\triangle = 1 + \frac{K_1}{s} + \frac{K_1K_2}{s}$$

$$\frac{\text{Char. Eqn.}}{CD\triangle} = \frac{s + (K_1)(1+K_2)}{s}$$

$$\text{Char. Eqn.} = s + K_1(1+K_2) = 0$$

Table V-1 gives an Order Number of (0,-I) for this compensator. Adding this to the proper element of the Location Matrix gives:

Fed from B to b: (Coef. Ind.) = (0,-I)

Fed from A to a: (Coef. Ind.) = (0,0)

Since the Characteristic Equation contains no negative powers of s, the first indicator (for B to b) appears to be contradictory. It is this very contradiction that yields the information on equation order. The order of the Characteristic Equation must be raised

by the amount that the "R" of the Coefficient Indicator is negative. That this occurred is borne out by comparison of the Characteristic Equations in the above example before and after this compensation. In the case of feedback from A to a, the "R" of the Coefficient Indicator accurately reflects that there is no change in order.

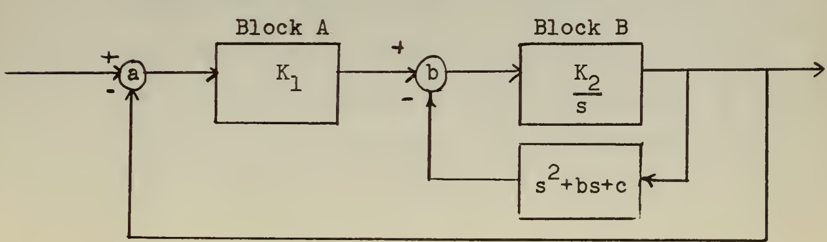
Rule I - The order of the Characteristic Equation is raised by the amount that the Roman Numeral of the Coefficient Indicator is negative.

Case II - Another order raising situation arises when any of the terms of

$$1 + \sum_r L_r + \sum_r \prod_2 L_r + \sum_r \prod_3 L_r + ---$$

when placed over a lowest common denominator, has a numerator of higher order than the uncompensated equation. Again, this is apparent if the Characteristic Equation is derived from the system performance function obtained by application of Mason's Rule.[11]

For example, if the sample system is compensated as follows:



$$\text{then, } \Delta = \begin{bmatrix} 1 & \frac{K_2}{s} \\ & \\ -K_1 & 1+ \frac{K_2(s^2+bs+c)}{s} \end{bmatrix}$$

$$\Delta = 1 + \frac{K_1K_2}{s} + \frac{K_2(s^2+bs+c)}{s}$$

$$\frac{\text{Char. Eqn.}}{CD\Delta} = \frac{K_2s^2 + (K_2b+1)s + (K_2c + K_1K_2}{s}$$

$$\text{Char. Eqn.} = s^2 + \left(\frac{K_2b+1}{K_2}\right)s + (K_1+c) = 0$$

Adding the Order Number of this compensator, (2,0), to the proper Location Matrix element gives a Coefficient Indicator of (2,0). Again there is an apparent contradiction in that the highest coefficient indicated to be changed is not present in the uncompensated Characteristic Equation. This contradiction indicates that the order of the Characteristic Equation must be raised.

Putting this compensator in the path from the output of Block A to Node a yields:

$$\Delta = \frac{\text{Char. Eqn.}}{CD\Delta} = \frac{K_1s^3 + K_1bs^2 + (K_1c+1)s + K_1K_2}{s}$$

$$\text{Char. Eqn.} = K_1s^3 + K_1bs^2 + (K_1c+1)s + K_1K_2 = 0$$

The Coefficient Indicator in this case is (2,I), which again indicates a raise in the order of the Characteristic Equation, the coefficients to be changed being a_3, a_2 and a_1.

Rule II - The order of the Characteristic Equation is raised by the amount

$$(A+R)_{\text{Coef. Ind.}} - (A+R)_{\text{Order of } CD_{\triangle}}$$

Note: If an occasion arises where both Rules I and II apply, first raise the Order of $CD_{\triangle}$ by the amount indicated in Rule I, put "R" of the Coefficient Indicator to zero, then, if necessary, apply Rule II.

Case III - A lag filter will always introduce its denominator in $CD_{\triangle}$. This will raise the order of the common denominator by an amount equal to the order of the filter. Since in practical servos, the order of the Characteristic Equation equals the order of the common denominator of $\triangle$, it too will be raised by this amount.

Rule III - The order of the Characteristic Equation is raised by the amount the Arabic Numeral of (Order of the Compensator) is negative.

5.3.3 <u>Determination of Quantitative Changes to Coefficients</u>

The Modified Characteristic Equation, Equation (5.3), consists of two terms:

$$\text{Mod. Char. Eqn.} = \frac{CD_{\triangle m}}{CD_{\triangle}}\,(\text{Char. Eqn.}) + G_{ij}\triangle_{ij}CD_{\triangle m}$$

This indicates that there are two contributions to the change in the coefficients. The first term's effect is predictable in the case of simple compensators. It is the amount all

coefficients are changed when the order of the Characteristic Equation is raised by applying Rules I and III above.

In Case I, $\frac{CD_{\triangle m}}{CD_{\triangle}} = s^{|R|}$ where the "R" is the Roman Numeral of the Coefficient Indicator.

In Case III, $\frac{CD_{\triangle m}}{CD_{\triangle}} = (s+a)$ or (s^2+bs+c) depending on the order of the filter used.

In these cases, the initial effect on coefficients can be readily calculated.

The change to coefficients a_{A+R}, a_{A-1+R}, etc., as indicated by the Coefficient Indicator can only be found by evaluating $G_{ij}\triangle_{ij}CD_{\triangle m}$ for the path concerned. Evaluating the change on coefficients caused by raising the order of the Characteristic Equation in accordance with Rule II can only be determined by evaluating $G_{ij}\triangle_{ij}CD_{\triangle m}$.

CHAPTFR 6

SELECTION OF COMPENSATOR

1. Introduction

The problem to be considered in this chapter is that of choosing the combination of compensator and location that can most easily change the characteristic equation of the system to one established by the specifications. This requires a comparison of the coefficients of the uncompensated characteristic equation to those of a minimum model equation. The actual coefficients which exceed a minimum allowable value will be used to select a working model equation. The working model equation permits calculation of a compensated characteristic equation consistent with the specifications.

There are four equations of interest in this chapter. The uncompensated characteristic equation describes the servomechanism before compensation. Its coefficients are set by the basic hardware. It is these coefficients which must be changed. The minimum model equation is the characteristic equation of a system with a transient response which has:

a. A specified dominant damped oscillatory mode,

b. A minimum number of first order modes, (none or one),

c. Non-dominant undamped oscillatory modes.

No first order modes will exist for even ordered systems. No

non-dominant oscillatory mode will exist for a third order system. Its coefficients are those of the characteristic equation of a system verging on instability. The working model equation is the characteristic equation of a system with a transient response which has a specified dominant damped oscillatory mode. Its non-dominant modes will be damped oscillatory and/or first order. Its coefficients are those of a characteristic equation of a stable system. The compensated characteristic equation describes the servomechanism after compensation. Depending on its stage of development, its coefficients may be purely numerical or may contain unspecified parameters. Compensation which changes the coefficients of the uncompensated equations to those of the compensated equation will result in a system with the specified response.

2. Choosing the Minimum Model Equation

There are three factors which determine the choice of a minimum model equation. They are:

1. It must be of the same order as the uncompensated characteristic equation.

2. The quadratic made up of the specified ζ and ω_n must be a factor of the model.

3. The dominance factors, α's, must be sufficiently large to insure that the response of the model has the desired correspondence with the response of the quadratic. The minimum model equations in Appendix B are so constructed that substitution of the desired ζ, ω_n and α's will give them the properties listed above.

3. Determination of the Compensated Characteristic Equation

The compensated characteristic equation must have coefficients at least as large as those of the minimum model. If its coefficients are less than those of the minimum model, the system must be unstable. A comparison of the uncompensated characteristic equation to the minimum model equation will show which coefficients must be changed. There are two possible conditions arising from this comparison. Either all of the coefficients are below the minimum, or a number of them are already sufficiently large. In the first case substitution of non-dominant damping ratios and/or acceptable dominance factors into the working models of the proper order, as given in Appendix B, will set the compensated characteristic equation. The ratio of a_1 to a_0 in the compensated characteristic equation may be set by static specifications. This may limit the choice of α's and non-dominant ζ's.

In the case where some or all of the coefficients are already above the minimum, several factors must be considered. These are:

1. The interdependence of the coefficients of the model equation.

2. The position of the satisfactory coefficients.

3. The effect of static specifications.

The interdependence of coefficients is expressed in the working model equations of Appendix B. In an n^{th} order model the n coefficients are functions of the specified ζ and ω_n, and of (n-2) dominance factors and/or non-dominant ζ's. This gives rise to n equations in n unknowns. Two of these

unknowns, ζ and ω_n of the dominant quadratic, are set by specifications. The remaining (n-2) degrees of freedom may be constrained by fixing either coefficients or α's. If any (n-2) coefficients of the uncompensated characteristic equation are above the minimum values, they will determine the required value of the remaining two coefficients. If fewer than (n-2) coefficients are satisfactory, the remaining degrees of freedom may be fixed by an arbitrary choice of α's. Required values for the remaining coefficients can then be calculated. These satisfactory and calculated coefficients comprise the compensated characteristic equation which will require minimum change in coefficients. The satisfactory coefficients should first be compared to the model which contains all real non-dominant roots. If they are not sufficiently large to fit this model, then the oscillatory working model equation must be used.

The second consideration arises from the fact that all compensators affect successive coefficients in the characteristic equation. As developed in Chapter 5, the independent coefficient changes are found by evaluating $G_{ij}\Delta_{ij}{}^{CD}/\Delta_m$. This will be in the form of $Ks^n(s+a)(s+b)$ etc. When expanded, it will contain successive powers of s. This fact is also revealed by the Coefficient Indicator. Because of this fact, the satisfactory coefficients used to determine the compensated characteristic equation should be chosen so that the coefficients remaining to be changed are successive.

The static specifications placed lower limits on some coefficients of the compensated characteristic equation.

These limits may force some coefficients to be larger than the dominance factors dictate. The coefficients so affected must be included in the (n-2) degree-of-freedom considerations above.

Comparison of the desired Coefficient Indicator and the Location Matrix will determine the type of compensator required to change the necessary coefficients. If it raises the order of the equation, the above procedure must be repeated starting with the minimum model equation of the higher order. The uncompensated equation to be compared with the model can be formed knowing the order raising effects described in Chapter 5.

4. Evaluating the Parameters of the Compensator

The $G_{ij}\Delta_{ij}$ of the required compensator is added to the old characteristic determinant. This results in a characteristic equation in which the coefficients requiring modification contain the parameters of the compensator. Equating these coefficients to those of the compensated characteristic equation will determine the value of the parameters. The compensator must have sufficient degrees of freedom to match the independent coefficient changes required.

CHAPTER 7

COMPENSATION DESIGN PROCEDURE

7.1 Introduction

The advantages of the technique described in the previous chapters are three. First, it involves no graphic presentations as required by other methods. Second, it does not depend on the intuition of the designer for its choice of compensation to achieve the desired result. Finally, it requires little of the extensive algebra inherent in other analytical methods. Since there is no intuition or experimental graphing involved, an ordered analytical approach to any compensation problem is possible.

This method divides naturally into four phases, which are the subjects of Chapters 3 through 6. These chapter divisions were made primarily in order to explain the technique. They also follow in the proper logical order for application to the solution of a problem. The step-by-step procedure in this chapter thus serves as a summary of the paper as well as a guide for the compensation designer.

7.2 Step-by-step Guide to Compensation

Given: I. A servomechanism for which the direct path components have been selected. This servo may have been partially compensated or may just have unity feedback from output to input.

II. Static and Dynamic Performance Specifications to be met by the system.

Problem: To design compensation to meet the specifications.

Solution:

Phase I: Forming the characteristic equation.

A. Arrange the system in a standard block diagram.

COMMENT: Combine the components between usable pickoffs and nodes into single blocks.

B. Form the characteristic determinant.

C. Write the characteristic equation by setting the expanded characteristic determinant equal to zero.

COMMENT: Note the Order Number of CD_{Δ}.

Phase II: Relating Specifications to Coefficients of Model Characteristic Equations.

A. Express static specifications in terms of restrictions on a_0 and a_1 of the compensated characteristic equation.

B. Interpret dynamic specifications in terms of second order response characteristics; i.e., ζ and ω_n.

C. Choose minimum acceptable dominance factors (α's and non-dominant ζ's) to insure second order dominance.

Phase III: Determining the Effect of Possible Compensators on Coefficients of the Uncompensated Characteristic Equation; i.e., Forming the Location Matrix.

COMMENT: Chapter 5 shows the details of this phase.

Phase IV: Choosing the Compensator.

A. Form the minimum model equation using α's, ζ's and ω_n found in Phase II.

COMMENT: All coefficients of the compensated equation must be greater than those of the minimum model to insure stability.

B. Choose the Working Model Equation.

1. Insure compliance with dynamic specifications. Put sufficiently large α's (Phase II-C) and the required ζ and ω_n (Phase II-B) in the working model equations of the proper order.

2. Insure compliance with static specifications. Compare a_1/a_o ratio, of the various forms of the working model equation. Choose for a working model that form which, with allowable non-dominant zetas, (Phase II-C) has the smallest coefficients that meet the specifications.

3. Remove the effect of static specifications from working model equation. In the event that the minimum model equation will not meet static specifications, the basic system configuration must be changed by addition of a component which reduces following error to zero. This will invariably mean the addition of an integration. Compensation of this type will introduce a zero in the system performance function. Set this zero to a value which will remove its effect on the dominant mode of the response (See Figure A-2). The design procedure must be started again with the revised system.

C. Form the compensated Characteristic equation.

1. Compare the uncompensated characteristic equation to the working model.

2. If n-m-2 coefficients of the uncompensated equation are above the working model coefficients and leave successive coefficients to be altered, use them to fill some of the n-2 degrees of freedom. Set m arbitrary α's or non-dominant ζ's. Compute the m+2 remaining coefficients of the compensated characteristic equation. If no coefficients are above those of the working model, it is the compensated characteristic equation.

D. Select the Compensator.

1. Form the desired Coefficient Indicator from the coefficients in C-2 above which must be changed.

2. Compare this Coefficient Indicator to the Location Matrix to find the simplest possible compensator. The compensator selected must introduce enough additional variable parameters so that the total number equals the number of coefficients to be changed plus the number of zeros of the system to be set (See COMMENT)

3. Form $\dfrac{G_{In}G_{jJ}\Delta_{ij}}{\Delta_m}$, which is the system performance function relating the actuating input at node "n" to the putput of block G_{jJ}. This expression will contain the adjustable parameters of the compensator. The compensated characteristic

equation will be the denominator of this function.

4. Equate the parametric form of the coefficients of the denominator of the system function to the coefficients of the compensated characteristic equation. Also set the parameters of any numerator zeros to a value which gives a satisfactory zero (i.e., sufficiently large α).

COMMENT: If the only possible compensator raises the order of the characteristic equation, apply the order raising rules of Chapter 5 to the uncompensated characteristic equation. This order raising will put a zero in the system function. Repeat phase III using higher order model.

CHAPTER 8

SAMPLE PROBLEM

8.1 Introduction

The problem to be worked in this chapter is a modification of an analysis problem presented by Chestnut and Mayer.[16] The problem, as presented in Chapter 13 of the reference, consisted of determining the performance of a position control system subjected to a load disturbance. Compensation had already been accomplished. It consisted of feedback of the first and second derivatives of the output quantity. The signal fed back was modified by a filter of the form $\frac{Ks^2(s+a)}{(s+b)(s+c)(s+d)}$.

The specifications were estimated from the performance of the system as compensated in the reference.

8.2 Statement of the Problem

Given: I The system to be compensated has the primary requirement of position control. However, it must respond properly, within specified limits, when subjected to a designated disturbance load torque. The main path components consist of the following:

Component	Transfer Function
Preamplifier	K_1 not specified
Amplifier, Amplidyne	$\frac{1500}{(s+31.5)}$ $\frac{850}{(s+17.6)}$
Motor and Load	$\frac{4.78}{s(s+9.1)}$
Disturbance Load/Voltage Transducer	K_T = .855 volt/ft.lb.

II Static Specifications

Steady State Error in response to step = 0.0

Following Error in response to ramp = 10^{-4} rad/rad/sec

Maximum output in response to load torque = 10^{-4} rad/ft.lb.

Dynamic Specifications:

Peak Overshoot = 1.2

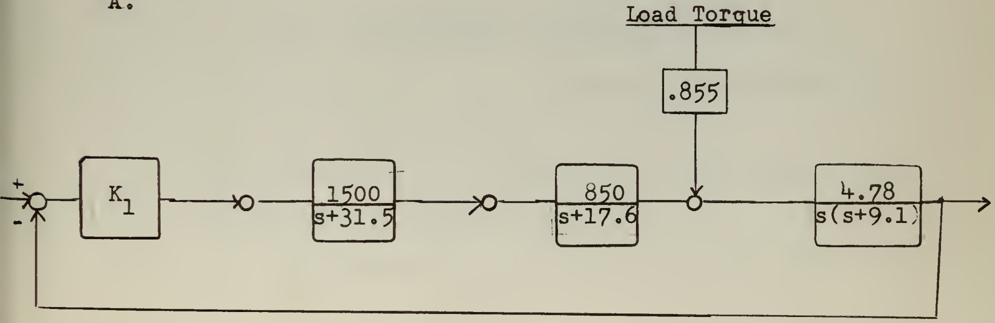
Settling Time to 1°/o of steady state in response to step = .6 sec

Problem: Design compensation such that the system will conform to specifications.

8.3 Solution of the Problem

Phase I: Forming the characteristic equation.

A.



B.

$$\triangle = \begin{vmatrix} 1 & 0 & 0 & \frac{4.78}{s(s+9.1)} \\ -K_1 & 1 & 0 & 0 \\ 0 & \frac{-1500}{s+31.5} & 1 & 0 \\ 0 & 0 & \frac{-850}{s+17.6} & 1 \end{vmatrix}$$

C.

$$\triangle = 1 + \frac{(6.09 \times 10^6)\, K_1}{s(s+31.5)(s+17.6)(s+9.1)}$$

$$\triangle = \frac{s(s+31.5)(s+17.6)(s+9.1)+(6.09 \times 10^6)\, K_1}{s(s+31.5)(s+17.6)(s+9.1)}$$

$$\triangle = \frac{s^4+58.2s^3+1001s^2+5040s+(6.09 \times 10^6)\, K_1}{s(s+31.5)(s+17.6)(s+9.1)}$$

Char. Eqn: $s^4 + 58.2s^3 + 1001s^2 + 5040s + (6.09 \times 10^6)K_1 = 0$

(Order of $CD_{\triangle}$) = (3,I)

Phase II: Relating Specifications to Coefficients of Model Characteristic Equation.

A. Static Specifications

1. Steady State Position Error in response to step = 0,

$$a_o = \underset{s \to 0}{\text{Limit}} \left[G_{dD} \triangle_{14} CD_{\triangle} \right]$$

$$a_o = \underset{s \to 0}{\text{Limit}} \left[\frac{4.78}{s(s+9.1)} \; \frac{(1.275 \times 10^6)K_1}{(s+31.5)(s+17.6)} \times s(s+9.1)(s+31.5)(s+17.6) \right]$$

$$a_o = (6.09 \times 10^6)K_1$$

The value of a_o is independent of this specification because of the unity feedback.

2. Following Error in response to ramp = 10^{-4} sec.

$$\text{Error}_{ss} = 10^{-4} > \frac{5040}{(6.09 \times 10^6)K_1}$$

$$K_1 > \frac{5040}{6.09 \times 10^2} = 8.28 \quad \text{or} \quad \frac{a_1}{a_o} < 10^{-4}$$

3. Response to Load Torque = 10^{-4} rad/ft.lb.

$$k > \lim_{s \to 0} \left[\frac{G_{IId} \Delta_{44} CD \Delta}{a_o} \right]$$

$$10^{-4} > \lim_{s \to 0} \left[\frac{(.855)(4.78)(s)(s+17.6)(s+31.5)(s+9.1)}{(6.09 \times 10^6)(s)(s+9.1)K_1} \right]$$

$$10^{-4} > \frac{(.855)(4.78)(17.6)(31.5)}{(6.09 \times 10^6)K_1}$$

$$K_1 > 3.43$$

B. Dynamic Specifications.

From reference 13, Peak Overshoot of 1.2 corresponds to ($.6 > \zeta > .5$). We will choose the conservative $\zeta = .6$.

From reference 10, Error is reduced to within $1^{\circ}/_{\circ}$ of final steady state value in four time constants, and

$$\text{Settling Time} = \frac{4}{\zeta\omega_n} = .6 \text{ sec.}$$

$$\omega_n = \frac{4}{.36} = 11.1 \text{ rad/sec}$$

C. Choosing Dominance Factors.

From Appendix A, $\alpha_1 \geq 3$ and $\alpha_2 \geq 2$ are acceptable values of dominance factors. If non-dominant roots are complex, use $\alpha \geq 3$ and $\zeta_2 \geq 3$.

Phase III: Determining the Effect of Possible Compensators.

$$\text{Auxiliary } \triangle = \begin{vmatrix} (0,0) & (-,-) & (-,-) & (-1,-I) \\ (-,-) & (0,0) & (-,-) & (-,-) \\ (-,-) & (-1,0) & (0,0) & (-,-) \\ (-,-) & (-,-) & (-1,0) & (0,0) \end{vmatrix}$$

$$\begin{matrix}\text{Order Numbers} \\ \text{of } \triangle_{ij}\text{'s}\end{matrix} = \begin{vmatrix} (0,0) & (0,0) & (-1,0) & (-2,0) \\ (-3,-I) & (0,0) & (-1,0) & (-2,0) \\ (-2,-I) & (-2,-I) & (0,0) & (-1,0) \\ (-1,-I) & (-1,-I) & (-2,-I) & (0,0) \end{vmatrix}$$

$$\begin{matrix}\text{Order Numbers} \\ \text{of } G_L\text{'s}\end{matrix} = \begin{vmatrix} (0,0) & (-1,0) & (-1,0) & (-1,I) \\ (0,0) & (-1,0) & (-1,0) & (-1,-I) \\ (0,0) & (0,0) & (-1,0) & (-1,-I) \\ (0,0) & (0,0) & (0,0) & (-1,-I) \end{vmatrix}$$

$$(\text{Order of } CD_{\triangle}) = (3,I)$$

$$\text{Location Matrix} = \begin{vmatrix} (3,I) & (2,I) & (1,I) & (0,0) \\ (0,0) & (2,I) & (1,I) & (0,0) \\ (1,0) & (1,0) & (2,I) & (1,0) \\ (2,0) & (2,0) & (1,0) & (2,0) \end{vmatrix}$$

Phase IV: Choosing the Compensator.

A. $a_3 = 2\zeta_1\omega_n \quad = \dfrac{\text{For } \alpha_1=3}{13.33}$

$$a_2 = (1+\alpha_1^{\,2})\omega_n^{\,2} = 1{,}233$$

$$a_1 = 2\zeta_1\alpha_1^{\,2}\omega_n^{\,3} = 14{,}080$$

$$a_o = a_1^2\omega_n^4 = 137,000$$

$$s^4 + 13.33s^3 + 1233s^2 + 14080s + 137000 = 0$$

Note that a_2 and a_1 of the uncompensated system are below minimum values.

B. $\frac{a_1}{a_o}$ for minimum model = .103

$\frac{a_1}{a_o}$ for all complex non-dominant roots = .126

$\frac{a_1}{a_o}$ for all real non-dominant roots = .156

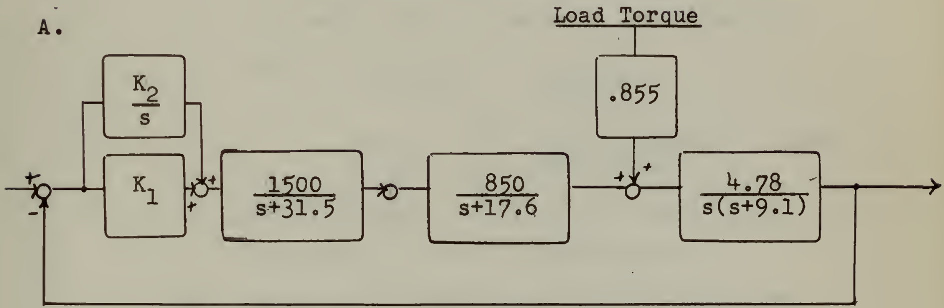
Since a_1/a_o from the model equations can never be less than .103, the condition on a_1/a_o imposed by static specifications must be relieved by making the following error independent of this ratio. This effect can be achieved by compensating so as to have zero following error. This implies making the servo a velocity control system. However, it may not lose its position control ability. Paralleling the main path with another forward path containing an extra integration will result in both position and velocity control. This effect can also be obtained by placing the extra integration in a feed forward path from the first to second nodes thereby utilizing succeeding integrations in both forward paths simultaneously. Using this compensation, the transfer function, G_{aA}, becomes $\frac{K_1(s + K_2/K_1)}{s}$

In addition to raising the order of the characteristic equation, a zero of the system performance function will be introduced at $s = -K_2/K_1$. Set this zero such that its

$a > 2$, say $K_2/K_1 = 25$. It is now necessary to recommence the solution.

Phase I: Forming the characteristic equation.

A.



B.

$$\Delta = \begin{vmatrix} 1 & 0 & 0 & \frac{4.78}{s(s+9.1)} \\ -(K_1+\frac{K_2}{s}) & 1 & 0 & 0 \\ 0 & -\frac{1500}{s+31.5} & 1 & 0 \\ 0 & 0 & -\frac{850}{s+17.6} & 1 \end{vmatrix}$$

C. $$\Delta = \frac{s^5 + 58.2s^4 + 1001s^3 + 5040s^2 + 6.09 \times 10^6 K_1 s + 6.09 \times 10^6 K_2}{s^2(s+9.1)(s+17.6)(s+31.5)}$$

$$\text{Char.Eqn.} = s^5+58.2s^4+1001s^3+5040s^2+6.09 \times 10^6 K_1 s+6.09 \times 10^6 K_2=0$$

(Order of CD_Δ) = (3,II)

Phase II: Relating Specifications to Coefficients of Model Characteristic Equations.

A. Statis Specifications:

1. Steady State Position Error in response to a step = 0

$$a_o = \lim_{s \to 0} \left[G_{dD} \Delta_{14} CD_{\Delta} \right] = 6.09 \times 10^6 K_2$$

As before, the value of a_o is independent of this specification.

2. Following Error in response to a ramp = 0

$$\text{Error} = \lim_{s \to 0} \frac{1}{s} \left[1 - \frac{G_{ii}\Delta_{ii}}{\Delta} \right]$$

$$= \lim_{s \to 0} \frac{s^4 + 58.2s^3 + 1001s^2 + 5040s}{\text{Char. Eqn.}}$$

Therefore, the values of the coefficients are independent of this specification.

3. Response to Load Torque < 10^{-4} rad/ft.lb.

$$10^{-4} > \lim_{s \to 0} \left[\frac{G_{IId} \Delta_{44} CD_{\Delta}}{\text{Char. Eqn.}} \right]$$

$$10^{-4} > \lim_{s \to 0} \frac{(.855)(4.78)(s)(s+17.6)(s+31.5)}{\text{Char. Eqn.}}$$

$$10^{-4} > 0$$

Therefore, the specification on load torque disturbance is met.

B. Dynamic Specifications:

Same ζ and ω_n are required.

C. The same dominance factors are acceptable.

Phase III: Determining the Effect of Possible Compensators.

The (Order of G_L) for the G_{Aa} feedback position is changed to (1,-I) by the addition of the parallel integration. The (Order of CD_{Δ}) is changed to (3,II).

$$\text{New Location Matrix} \quad \begin{bmatrix} (2,I) & (2,II) & (1,II) & (0,I) \\ (0,I) & (2,II) & (1,II) & (0,I) \\ (1,I) & (1,I) & (2,II) & (1,I) \\ (2,I) & (2,I) & (1,I) & (2,I) \end{bmatrix}$$

Phase IV: Choosing the Compensator.

A. Using $\alpha_1 = 3$ and $\alpha_2 = 2$, the minimum fifth order model is:

$$s^5 + 35.53s^4 + 1527s^3 + 42,100s^2 + 464,000s + 3,030,000=0$$

Since this is now a Type II system, it has zero error in response to inputs of position and velocity. The a_1/a_o ratio represents velocity error in response to an acceleration input.

The requirement that the working model meet static specifications is removed.

B. 1. Since no working model had to be formed to consider the static specifications, the initial comparison will be made to the minimum model. With any easily realizable values for K_1 and K_2, the a_4, a_1 and a_o of the uncompensated equation are above those of the minimum model. Since a_4 is the only fixed coefficient above the minimum model, it will be used as a satisfactory coefficient. K_1 and K_2 will be left unspecified.

2. n-m-2=1 and m=5-2-1=2 arbitrary dominance factors. Solving the parametric form of the a_4 coefficient of the oscillatory working model, with $\alpha_1=3$ and $\alpha_2=2$, yields:

$$(2\zeta_1 + 2\alpha_1\zeta_2 + \alpha_2)\omega_n = 58.2$$

$$(1.2 + 6\zeta_2 + 2)11.1 = 58.2$$

$$\zeta_2 = .534$$

This non-dominant damping ratio is larger than needed, so $\zeta_2 = .3$ can be used to resolve for a larger α_2.

$$(1.2 + 1.8 + \alpha_2)11.1 = 58.2$$
$$\alpha_2 = 2.24$$

Using these values for α's and ζ's, the remaining coefficients can be found.

$a_3 = 2820$

$a_2 = 54600$

$a_1 = 564000$

$a_o = 3415000$

The compensated characteristic equation is:

$$s^5 + 58.2s^4 + 2820s^3 + 54600s^2 + 564000s + 3415000 = 0$$

C. 1. The coefficients to be changed are a_3, a_2, a_1 and a_o. a_1 and a_o could be changed independently of additional compensation by adjusting K_1 and K_2, but K_2/K_1 has been set at 25. Therefore it is necessary that compensation provide for varying a_1 and a_o independently of the other. In order to keep the coefficients to be changed successive, K_2 will be determined. From the characteristic equation developed in Phase I-C and the compensated equation developed in Phase IV-C-2:

$$6.09 \times 10^6\ K_2 = 3{,}415{,}000$$
$$K_2 = .56$$

Therefore, $K_1 = K_2/25 = .0224$

The "semi-compensated" characteristic equation is now:

$$s^5 + 58.2s^4 + 1001s^3 + 5040s^2 + 137{,}000s + 3{,}415{,}000=0$$

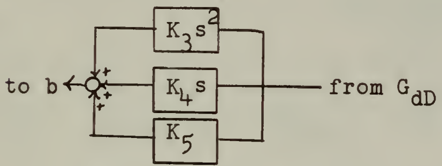
The compensated characteristic equation is:

$$s^5 + 58.2s^4 + 2820s^3 + 54{,}600s^2 + 564{,}000s + 3{,}415{,}000=0$$

Comparison of these two equations gives the desired Coefficient Indicator to be (2,I).

2. The compensator should be the simplest one which has sufficient adjustable parameters to change the required number of coefficients. The compensator must have three adjustable parameters since there are three coefficients to be raised.

The compensator must be in a feedback path since feed forward compensators reduce the size of the coefficients they effect. Comparison of the desired Coefficient Indicator and the Location Matrix indicates that feedback position G_{Dd} requires the simplest compensator with necessary parameters, one with order number (2,0). This can be achieved with a transfer function of the form K(s+a)(s+b) which will arise from the following:



$$PF = K_3\left(s^2 + {}^{K_4}/_{K_3}\, s + {}^{K_5}/_{K_3}\right)$$

3. $\Delta_m = \Delta + G_{ij}\Delta_{ij}$

$$\Delta_m = \frac{s^5+58.2s^4+1001s^3+5040s^2+137{,}000s+3{,}415{,}000}{s^2(s+9.1)(s+17.6)(s+31.5)}$$

$$+ \frac{(4.78)(1500)(850)K_3(s^2+{}^{K_4}/K_3\, s+{}^{K_5}/K_3)}{s(s+9.1)(s+17.6)(s+31.5)}$$

$$G_{Ia}G_{dD}\Delta_{14} = (1)\left(\frac{4.78}{s(s+9.1)}\right)\left(\frac{.0224(s+25)}{s}\;\frac{850}{s+17.6}\;\frac{1500}{s+31.5}\right)$$

$$\frac{G_{Ia}G_{dD}\Delta_{14}}{\Delta_m} = \frac{137,000(s+25)}{s^5+58.2s^4+(6.09\text{x}10^6K_3+1001)s^3+(6.09\text{x}10^6K_4+5040)s^2+(6.09\text{x}10^6K_5)s+3,415,000}$$

Equating coefficients of the two forms of the compensated equations give the following relationships:

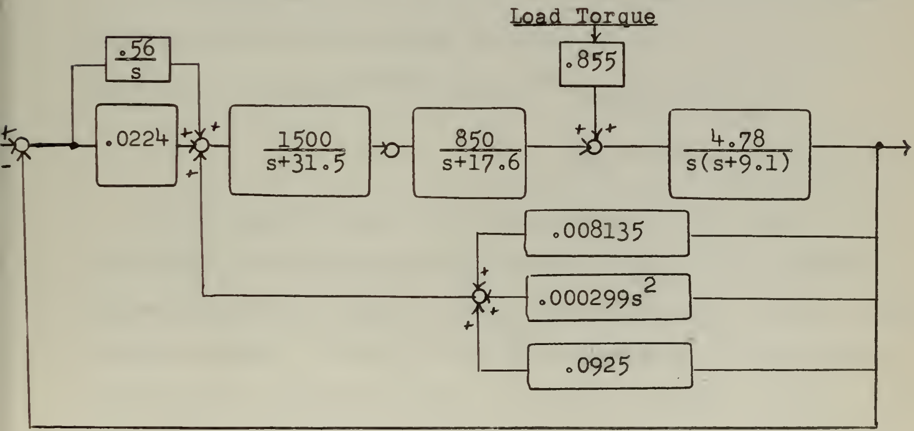
$$a_3 = 6.09 \text{ x } 10^6\ K_3 + 1001 = 2820 \quad \text{or } K_3 = .000299$$

$$a_2 = 6.09 \text{ x } 10^6\ K_4 + 5040 = 54,600 \quad \text{or } K_4 = .00813$$

$$a_1 = 6.09 \text{ x } 10^6\ K_5 = 564,000 \quad \text{or } K_5 = .0925$$

8.4 Checking the Solution

The compensated system has the following block diagram:



$$\Delta_m = \begin{vmatrix} 1 & 0 & 0 & \dfrac{4.78}{s(s+9.1)} \\ -(.0224+\dfrac{.56}{s}) & 1 & 0 & \left[\dfrac{4.78}{s(s+9.1)}\right](.299\times10^{-3}s^2 + .813\times10^{-2}s+.0925) \\ 0 & -\dfrac{1500}{s+31.5} & 1 & 0 \\ 0 & 0 & -\dfrac{850}{s+17.6} & 1 \end{vmatrix}$$

$$\Delta_m = \frac{s^5+58.2s^4+2820s^3+54{,}600s^2+564{,}000s + 3{,}415{,}000}{s^2(s+9.1)(s+31.5)(s+17.6)}$$

For an input of a unit step to node a and output from block G_{dD}, the system performance function is:

$$PF = \frac{137{,}000(s+25)}{s(s+24.9)(s^2+20s+1109)(s^2+13.3s+123.2)}$$

The response of the system in time domain can be found by the method of evaluating residues to be:

$$\theta_{out}(t) = 1+.0011e^{-24.9t}+.137e^{-10t}\cos(31.8t-30^\circ.6) +1.48e^{-6.66t}\cos(8.88t-139^\circ.5)$$

It is apparent that the contribution of the real root is negligible, being only on the order of $1^o/_o$ at its greatest. The contribution of the non-dominant oscillatory roots, though somewhat larger, is still less than one-tenth of the contribution of the dominant roots. It decays quickly and by t = .2 adds only about $2^o/_o$ to the value of the output. At t = .6 sec, the value of the output is about .98, a $2^o/_o$ deviation from final value. A $1^o/_o$ deviation was specified. At t = .27,

peak overshoot occurs and equals 1.24; compared to the specified 1.20. Failure to meet specifications exactly in these two instances is attributable to placing the zero too close to the origin. This indicates a need for further study of the effect of zeros upon the transient response, or for a more conservative choice of α's in connection with zeros.

The solution obviously meets static specifications because of the introduction of the feed forward integration compensation.

8.5 Summary of the Application of the Technique to this Problem

The solution of the problem proceeded along straightforward lines until it was discovered that a Position Control System could not provide acceptable following error. It was then necessary to stipulate compensation which would give the system Velocity Control characteristics. With this compensation inserted, the solution was then recommenced. The necessity of another compensator became apparent and its characteristics were determined.

It is interesting to note the similarity between the compensation determined by this solution and that provided by Chestnut and Mayer.[16] Both involve feedback of first and second derivatives of the output. However, this solution replaces their elaborate filter, $\frac{Ks^2(s+a)}{(s+b)(s+c)(s+d)}$, with feed forward through the function $\frac{K}{s}$ and feedback of a signal proportional to the output quantity.

CHAPTER 9

CONCLUSIONS AND RECOMMENDATIONS

9.1 Conclusions. Application of step-by-step procedures developed in this thesis resulted in successfully designing compensation for a system which, though not overly complex, was subject to somewhat severe specifications. Alternate methods of compensation design may have resulted in equally simple comepnsation but, by their trial-and-error nature, must be assumed to be less straightforward. It it therefore concluded that the proposed compensation technique is valid and presents the designer with a powerful and labor-saving tool.

9.2 Recommendations. Several areas for further study present themselves upon consideration of the results of this thesis and of the steps by which these results were obtained. Specific areas requiring further work are:

a. More exact determination of the deviation from second-order response caused by the existence of zeros and additional roots.

b. Careful consideration of the step-by-step procedure to eliminate possible duplication of effort when order raising compensation is used.

Areas of research which should provide beneficial results are:

a. Examination of the means by which specifications can

be related to dominant third-order response.

b. Consideration of creating non-dominant oscillatory modes at harmonics of the dominant mode, in order to better reproduce the wave form of the input.

c. Examination of the means by which this method may be extended to include treatment of other than positional servomechanisms.

APPENDIX A

A DETAILED EXAMINATION OF THE EFFECT OF α ON THE DEVIATION OF A THIRD ORDER RESPONSE FROM THAT OF A SECOND ORDER

To examine the effect of α_1 on the deviation of the third order response from the second order, consider the variation of the two systems if a unit step is applied to each.

Second Order: $$\mathrm{Out}(s) = \frac{1}{s(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \text{(A.1)}$$

$$\mathrm{Out}(t) = \frac{1}{\omega_n^2} + \frac{1}{\sqrt{1-\zeta^2}\,\omega_n^2} e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\,\omega_n t+\psi_i) \qquad \text{(A.2)}$$

where ψ_i = Phase lag of ideal (2nd order) response =

$- \arctan \dfrac{\sqrt{1-\zeta^2}}{-\zeta}$

Third Order: $$\mathrm{Out}(s) = \frac{\alpha_1\omega_n}{s(s+\alpha_1\omega_n)(s^2+2\zeta\omega_n s+\omega_n^2)} \qquad \text{(A.3)}$$

where $\alpha_1\omega_n$ is included to make steady state values equal.

$$\mathrm{Out}(t) = \frac{1}{\omega_n^2} - \frac{e^{-\alpha\omega_n t}}{\omega_n^2[\alpha^2-2\zeta d+1]} +$$

$$\frac{\alpha_1 e^{-\zeta\omega_n t} \sin(\sqrt{1-\zeta^2}\,\omega_n t+\psi_a}{\omega_n^2[(1-\zeta^2)(\alpha^2-2\zeta\alpha+1)]^{1/2}} \qquad \text{(A.4)}$$

where ψ_a = Phase lag of actual (3rd order) response =

$$- \arctan \frac{\sqrt{1-\zeta^2}}{-\zeta} - \arctan \frac{\sqrt{1-\zeta^2}}{\alpha-\zeta} .$$

Define deviation as follows:

$$D = \frac{\text{actual - ideal}}{\text{ideal}} \tag{A.5}$$

Substitution of equations (A.2) and (A.4) into equation (A.5) yields:

$$D = \frac{\dfrac{-\sqrt{1-\zeta^2}\; e^{-(\alpha-\zeta)\omega_n t}}{[\alpha^2-2\zeta\alpha+1]} + \dfrac{\alpha \sin(\sqrt{1-\zeta^2}\;\omega_n t+\psi_a)}{[\alpha^2-2\zeta\alpha+1]} - \sin(\sqrt{1-\zeta^2}\;\omega_n t+\psi_i)}{\sqrt{1-\zeta^2}\; e^{-\zeta\omega_n t} + \sin(\sqrt{1-\zeta^2}\;\omega_n t + \psi_i)} \tag{A.6}$$

This deviation is a function of t, α, and ζ of the dominant roots. Ideally we would like to find the maximum deviation, no matter when it occurs. This, however, is a great undertaking which is beyond the scope of this paper. To simplify the problem we have picked the time corresponding to the peak overshoot of the second order response:

$$T_p = \frac{\frac{\pi}{2} + \arctan \frac{\sqrt{1-\zeta^2}}{-\zeta}}{\omega_n}$$

Substituting this time into equation (A.6):

$$D = \frac{\dfrac{-\sqrt{1-\zeta^2}\; e^{-(\alpha-\zeta)\omega_n T_p}}{[\alpha^2-2\zeta\alpha+1]} + \dfrac{\alpha \sin(\frac{\pi}{2}+\arctan\frac{\sqrt{1-\zeta^2}}{\alpha-\zeta}}{[\alpha^2-2\zeta\alpha+1]^{1/2}} -1}{\sqrt{1-\zeta^2}\; e^{j\omega_n T_p} + 1} \tag{A.7}$$

$$\text{but Sin } \left(\frac{\pi}{2}+\arctan\frac{\sqrt{1-\zeta^2}}{d-\zeta}\right) = \frac{\alpha-\zeta}{[\alpha^2-2\zeta\alpha+1)^{1/2}}$$

so:

$$D = \frac{-\sqrt{1-\zeta^2}\; e^{-(\alpha-\zeta)\omega_n T_p} + \alpha\zeta+1}{[1+\sqrt{1-\zeta^2}\; e^{-J\omega_n T_p}][\alpha^2-2\zeta\alpha+1]} \qquad (A.8)$$

This equation is plotted in figure A.1. Figure A.2 represents the result of similar consideration of the effect of a zero at $\alpha\omega_n$ on a second order response.

Examination of these curves will yield an appropriate value of α to insure the desired degree of dominance of the second order response.

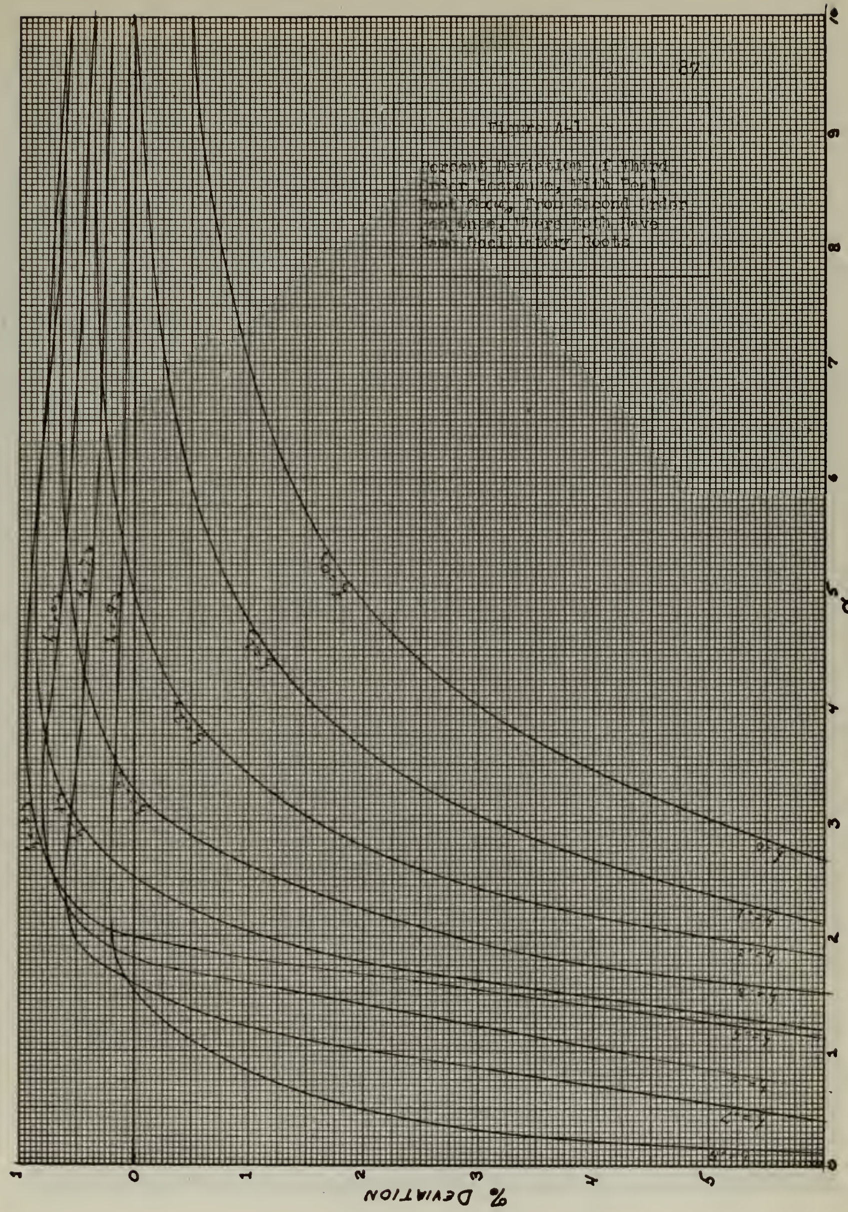


Figure A-1

Percent Deviation of Third Order Response, With Real Root $\alpha\omega_n$, From Second Order Response, Where Both Have Same Oscillatory Roots

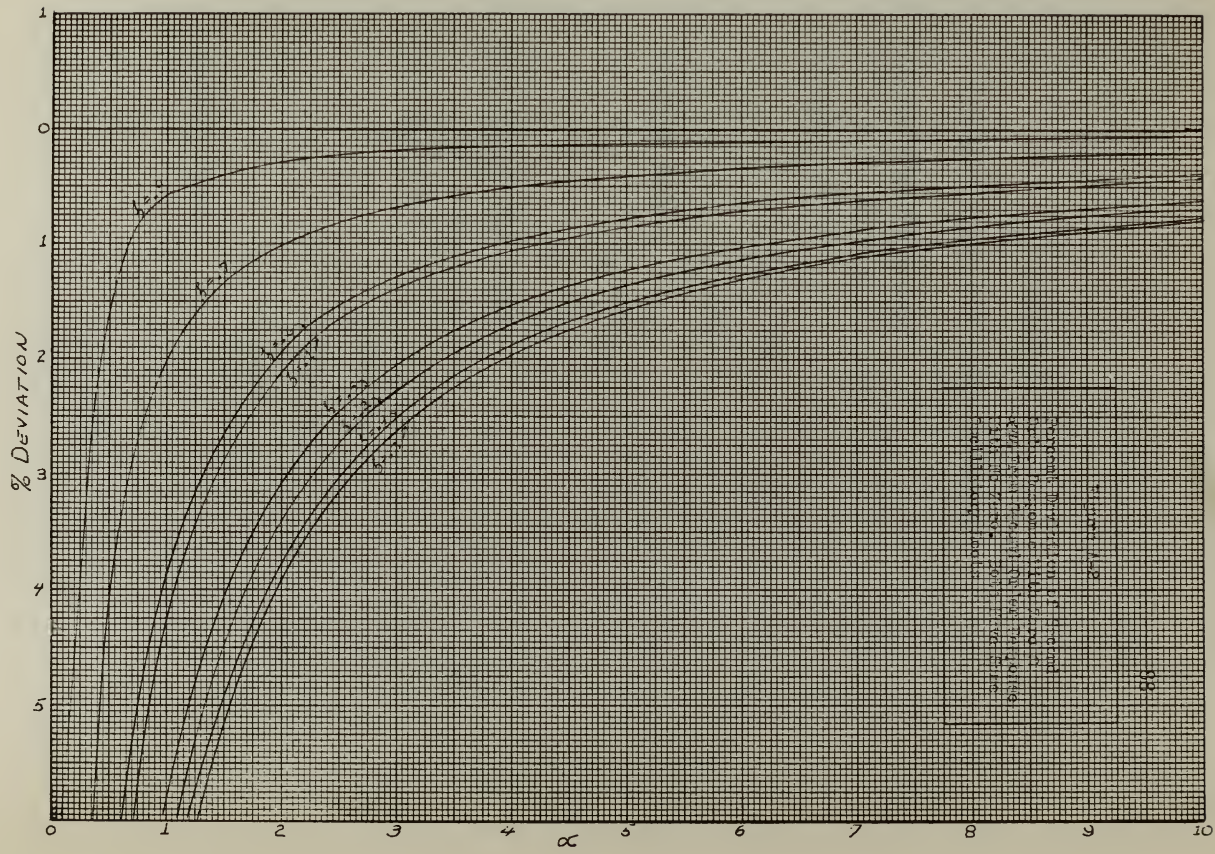


Figure A-2

Percent Deviation of Second Order Response With Zero With True Second Order Response With No Zero. Both Have Same Oscillatory Roots

APPENDIX B

THE MODEL CHARACTERISTIC EQUATIONS

B-1 Second-Order Equations

$$s^2 + a_1 s + a_o = 0$$

Minimum Model:

$$a_1 = 2\zeta\omega_n$$

$$a_o = \omega_n^2$$

Working Model:

$$a_1 = 2\zeta\omega_n$$

$$a_o = \omega_n^2$$

B-2 Third-Order Equations

$$s^3 + a_2 s^2 + a_1 s + a_o = 0$$

Minimum Model:

$$a_2 = (2\zeta+\alpha_1)\omega_n$$

$$a_1 = (1+2\zeta\alpha_1)\omega_n^2$$

$$a_o = (\alpha_1)\omega_n^3$$

Working Model:

$$a_2 = (2\zeta+\alpha_1)\omega_n$$

$$a_1 = (1+2\zeta\alpha_1)\omega_n^2$$

$$a_o = (\alpha_1)\omega_n^3$$

B-3 <u>Fourth-Order Equations</u>

$$s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0$$

Minimum Model:

$$a_3 = 2\zeta\omega_n$$

$$a_2 = (1+\alpha_1^{\,2})\omega_n^{\,2}$$

$$a_1 = (2\zeta\alpha_1^{\,2})\omega_n^{\,3}$$

$$a_o = (\alpha_1^{\,2})\omega_n^{\,4}$$

Working Model (Non-dominant roots complex)

$$a_3 = (2\zeta_1+2\alpha_1\zeta_2)\omega_n$$

$$a_2 = (1+4\alpha_1\zeta_1\zeta_2+\alpha_1^{\,2})\omega_n^{\,2}$$

$$a_1 = (2\alpha_1\zeta_2+2\zeta_1\alpha_1^{\,2})\omega_n^{\,3}$$

$$a_o = (\alpha_1^{\,2})\omega_n^{\,4}$$

Working Model (Non-dominant roots real)

$$a_3 = (2\zeta+\beta_1)\omega_n$$

$$a_2 = (1+2\zeta\beta_1+\beta_2)\omega_n^{\,2}$$

$$a_1 = (\beta_1+2\zeta\beta_2)\omega_n^{\,3}$$

$$a_o = (\beta_2)\omega_n^{\,4}$$

where

$$\beta_1 = \alpha_1+\alpha_2$$

$$\beta_2 = \alpha_1\alpha_2$$

B-4 <u>Fifth-Order Equations</u>

$$s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0$$

Minimum Model:

$$a_4 = (2\zeta_1+\alpha_2)\omega_n$$

B-4 Fifth Order Equations (cont'd)

Minimum Model (cont'd)

$$a_3 = (1+\alpha_1{}^2+2\zeta_1\alpha_2)\omega_n{}^2$$

$$a_2 = (2\zeta_1\alpha_1{}^2+\alpha_2+a_1{}^2\alpha_2)\omega_n{}^3$$

$$a_1 = (\alpha_1{}^2+2\zeta_1\alpha_1{}^2\alpha_2)\omega_n{}^4$$

$$a_o = (a_1{}^2\alpha_2)\omega_n{}^5$$

Working Model (One pair non-dominant roots complex)

$$a_4 = (2\zeta_1+2\alpha_1\zeta_2+\alpha_2)\omega_n$$

$$a_3 = (1+4\alpha_1\zeta_1\zeta_2+\alpha_1{}^2+2\zeta_1\alpha_2+2\alpha_1\alpha_2\zeta_2)\omega_n{}^2$$

$$a_2 = (2\alpha_1\zeta_2+2\zeta_1\alpha_1{}^2+\alpha_2+4\alpha_1\alpha_2\zeta_1\zeta_2+\alpha_1{}^2\alpha_2)\omega_n{}^3$$

$$a_1 = (\alpha_1{}^2+2\alpha_1\alpha_2\zeta_2+2\alpha_1{}^2\alpha_2\zeta_1)\omega_n{}^4$$

$$a_o = (\alpha_1{}^2\alpha_2)\omega_n{}^5$$

Working Model (Non-dominant roots real)

$$a_4 = (2\zeta_1+\beta_1)\omega_n$$

$$a_3 = (1+2\zeta_1\beta_1+\beta_2)\omega_n{}^2$$

$$a_2 = (\beta_1+2\zeta_1\beta_2+\beta_2)\omega_n{}^3$$

$$a_1 = (\beta_2+2\zeta_1\beta_3)\omega_n{}^4$$

$$a_o = (\beta_3)\omega_n{}^5$$

where

$$\beta_1 = \alpha_1 + \alpha_2 + \alpha_3$$

$$\beta_2 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$$

$$\beta_3 = a_1\alpha_2\alpha_3$$

B-5 <u>Sixth-Order Equations</u>:

$$s^6 + a_5 s^5 + a_4 s^4 + a_3 s^3 + a_2 s^2 + a_1 s + a_o = 0$$

Minimum Model:

$$a_5 = (2\zeta_1)\omega_n$$

$$a_4 = (1+\alpha_1^2+\alpha_2^2)\omega_n^2$$

$$a_3 = (2\zeta_1\alpha_1^2+2\zeta_1\alpha_2^2)\omega_n^3$$

$$a_2 = (\alpha_1^2+\alpha_2^2+\alpha_1^2\alpha_2^2)\omega_n^4$$

$$a_1 = (2\zeta_1\alpha_1^2\alpha_2^2)\omega_n^5$$

$$a_o = (\alpha_1^2\alpha_2^2)\omega_n^6$$

Working Model (Both pairs non-dominant roots complex)

$$a_5 = (2\zeta_1+2\alpha_1\zeta_2+2\alpha_2\zeta_3)\omega_n$$

$$a_4 = (1+4\alpha_1\zeta_1\zeta_2+\alpha_1^2+4\zeta_1\zeta_3\alpha_2+4\zeta_2\zeta_3\alpha_1\alpha_2+a_2^2)\omega_n^2$$

$$a_3 = (2\alpha_1\zeta_2+2\zeta_1\alpha_1^2+2\alpha_2\zeta_3+8\alpha_1\alpha_2\zeta_1\zeta_2\zeta_3+2\zeta_3\alpha_1^2\alpha_2$$
$$+2\zeta_1\alpha_2^2+2\alpha_1\alpha_2^2\zeta_2)\omega_n^3$$

$$a_2 = (\alpha_1^2+4\alpha_1\alpha_2\zeta_2\zeta_3+4\alpha_1^2\alpha_2\zeta_1\zeta_3+\alpha_2^2+4\zeta_1\zeta_2\alpha_1\alpha_2^2$$
$$+\alpha_1^2\alpha_2^2)\omega_n^4$$

$$a_1 = (2\zeta_3\alpha_1^2\alpha_2+2\alpha_1\alpha_2^2\zeta_2+2\alpha_1^2\alpha_2^2\zeta_1)\omega_n^5$$

$$a_o = (\alpha_1^2\alpha_2^2)\omega_n^6$$

Working Model (One pair non-dominant roots real)

$$a_5 = (2\zeta_1+2\alpha_1\zeta_2+\beta_1)\omega_n$$

$$a_4 = (1+4\alpha_1\zeta_1\zeta_2+\alpha_1^2+2\zeta_1\beta_1+2\zeta_2\alpha_1\beta_1+\beta_2)\omega_n^2$$

$$a_3 = (2\alpha_1\zeta_2+2\zeta_1\alpha_1^2+\beta_1+4\alpha_1\beta_1\zeta_1\zeta_2+\alpha_1^2\beta_1+2\zeta_1\beta_2$$
$$+2\alpha_1\beta_2\zeta_2)\omega_n^3$$

B-5 Sixth-Order Equations (cont'd):

Working Model (One pair non-dominant roots real)

$$a_2 = (\alpha_1{}^2+2\alpha_1\beta_1\zeta_2+2\zeta_1\alpha_1{}^2\beta_1+\beta_2+4\alpha_1\beta_2\zeta_1\zeta_2+\alpha_1{}^2\beta_2)\omega_n{}^4$$

$$a_1 = (\alpha_1{}^2\beta_1+2\alpha_1\beta_2\zeta_2+2\alpha_1{}^2\beta_2\zeta_1)\omega_n{}^5$$

$$a_o = (\alpha_1{}^2\beta_2)\omega_n{}^6$$

where

$$\beta_1 = \alpha_2+\alpha_3$$

$$\beta_2 = \alpha_2\alpha_3$$

Working Model (All non-dominant roots real)

$$a_5 = (2\zeta_1+\beta_1)\omega_n$$

$$a_4 = (1+2\zeta_1\beta_1+\beta_2)\omega_n{}^2$$

$$a_3 = (\beta_1+2\zeta_1\beta_2+\beta_3)\omega_n{}^3$$

$$a_2 = (\beta_2+2\zeta_1\beta_3+\beta_4)\omega_n{}^4$$

$$a_1 = (\beta_3+2\zeta_1\beta_4)\omega_n{}^5$$

$$a_o = (\beta_4)\omega_n{}^6$$

where

$$\beta_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4$$

$$\beta_2 = \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_3\alpha_4$$

$$\beta_3 = \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_3\alpha_4$$

$$\beta_4 = \alpha_1\alpha_2\alpha_3\alpha_4$$

B-6 Seventh-Order Equations:

$$s^7 + a_6s^6 + a_5s^5 + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_o = 0$$

Minimum Model:

$$a_6 = (2\zeta_1+\alpha_3)\omega_n$$

$$a_5 = (1+\alpha_1^2+\alpha_2^2+2\zeta_1\alpha_2)\omega_n^2$$

$$a_4 = (2\zeta_1\alpha_1^2+2\zeta_1\alpha_2^2+\alpha_3+\alpha_1^2\alpha_3+\alpha_2^2\alpha_3)\omega_n^3$$

$$a_3 = (\alpha_1^2+\alpha_2^2+\alpha_1^2\alpha_2^2+2\zeta_1\alpha_1^2\alpha_3+2\zeta_1\alpha_2^2\alpha_3)\omega_n^4$$

$$a_2 = (2\zeta_1\alpha_1^2\alpha_2^2+\alpha_1^2\alpha_3+\alpha_2^2\alpha_3+\alpha_1^2\alpha_2^2\alpha_3)\omega_n^5$$

$$a_1 = (\alpha_1^2\alpha_2^2+2\zeta_1\alpha_1^2\alpha_2^2\alpha_3)\omega_n^6$$

$$a_o = (\alpha_1^2\alpha_2^2\alpha_3)\omega_n^7$$

Working Model (Some roots complex)

$$a_6 = (A_5+\alpha)\omega_n$$

$$a_5 = (A_4+\alpha A_5)\omega_n$$

$$a_4 = (A_3+\alpha A_4)\omega_n$$

$$a_3 = (A_2+\alpha A_3)\omega_n$$

$$a_2 = (A_1+\alpha A_2)\omega_n$$

$$a_1 = (A_o+\alpha A_1)\omega_n$$

$$a_o = (\alpha A_o)\omega_n$$

where A_n refers to coefficients of same type model in sixth order eqns.

Working Model (All non-dominant roots real)

$$a_6 = (2\zeta_1+\beta_1)\omega_n$$

$$a_5 = (1+2\zeta_1\beta_1+\beta_2)\omega_n^2$$

$$a_4 = (\beta_1+2\zeta_1\beta_2+\beta_3)\omega_n^3$$

$$a_3 = (\beta_2+2\zeta_1\beta_3+\beta_4)\omega_n^4$$

$$a_2 = (\beta_3+2\zeta_1\beta_4+\beta_5)\omega_n{}^5$$

$$a_1 = (\beta_4+2\zeta_1\beta_5)\omega_n{}^6$$

$$a_o = (\beta_5)\omega_n{}^7$$

where

$$\beta_1 = \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5$$

$$\beta_2 = \alpha_1\alpha_2+\alpha_1\alpha_3+\alpha_1\alpha_4+\alpha_1\alpha_5+\alpha_2\alpha_3+\alpha_2\alpha_4+\alpha_2\alpha_5+\alpha_3\alpha_4+\alpha_3\alpha_5+\alpha_4\alpha_5$$

$$\beta_3 = \alpha_1\alpha_2\alpha_3+\alpha_1\alpha_2\alpha_4+\alpha_1\alpha_2\alpha_5+\alpha_2\alpha_3\alpha_4+\alpha_2\alpha_3\alpha_5+\alpha_3\alpha_4\alpha_5$$

$$+\alpha_1\alpha_3\alpha_4+\alpha_1\alpha_3\alpha_5+\alpha_1\alpha_4\alpha_5+\alpha_2\alpha_4\alpha_5$$

$$\beta_4 = \alpha_1\alpha_2\alpha_3\alpha_4+\alpha_1\alpha_2\alpha_3\alpha_5+\alpha_2\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_3\alpha_4\alpha_5+\alpha_1\alpha_2\alpha_4\alpha_5$$

$$\beta_5 = \alpha_1\alpha_2\alpha_3\alpha_4\alpha_5$$

APPENDIX C

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